DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION. ALEXANDRIA. VIRGINIA

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY
CALIFORNIA

NOTES ON FOURIER ANALYSIS AND SPECTRAL WINDOWS

BY
EMANUEL PARZEN

TECHNICAL REPORT NO. 48
MAY 15, 1963

NOTES ON FOURIER ANALYSIS AND SPECTRAL WINDOWS

by

Emanuel Parzen

TECHNICAL REPORT NO. 48

May 15, 1963

PREPARED UNDER CONTRACT Nonr-225(21)

(NR-042-993)
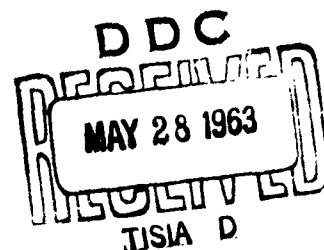
FOR

OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD   UNIVERSITY

STANFORD, CALIFORNIA

Notes On Fourier Analysis and Spectral Windows

by

Emanuel Parzen

Summary: This is an expository paper which seeks to systematically
develop some basic ideas about Fourier analysis and spectral windows in
order to have a convenient reference for results to be used in other
work by the author on statistical spectral analysis.

1. Introduction

The purpose of science, it is said, is "to understand the present,
to interpret the past, to predict the future." To achieve these aims, it
attempts to determine the relations between different physical quantities
which it expresses in the forms of theories (or "laws") abstracted from
observations.

In many scientific fields, the basic data are obtained in the form
of time series (that is, the data consist of a set of observations, each
associated with a particular point of time.) Thus if one is studying
temperature variations, one might begin by obtaining records of daily
mean temperatures at various stations. If one is studying death rates,
one might begin by obtaining records of annual mortality rates in various
age groups. Consequently, the problem of time series analysis plays a
role in all scientific fields.

In analyzing time series representing the behavior of physical
phenomena over time, one of the main approaches of researchers has been
to look for features which recur systematically after fixed periods of
time, so that their occurrence can be accurately foreseen. By the success-
ful application of such an approach, man is able to predict the tides
and the times of sunrise and sunset. Consequently, among the problems
which historically came to be considered in time series analysis were
the problems of detecting hidden periodicities and of measuring seasonal
variation.

These classical problems are nowadays being approached as part of
the overall problem of statistical spectral analysis. Statistical spectral
analysis may be defined as a theory which seeks to provide a method

whereby from observed records, of finite length, of an empirical phenomenon one can interpret the time functions representing the phenomenon as a superposition (finite or infinite linear combination) of harmonics cos ωt and sin ωt.

The aim of this paper is to systematically develop some basic ideas about Fourier analysis and spectral windows. The author hopes to show in later publications [see Parzen (1963)] the usefulness of these ideas in obtaining a practical solution to the problems of statistical spectral analysis.

The paper consists of five sections: (1) introduction, (2) the Fourier coefficients of a periodic functions, (3) windows, (4) coefficient windows, (5) evaluation of Fourier coefficients using equi-spaced points.

2. The Fourier coefficients of a periodic function.

One approach to the problem of measuring or representing a function
defined on an interval is to try to summarize the function $f(\cdot)$ by a
finite number of coefficients. Various methods of doing this are examined
in this section.

Periodic functions. A time function $f(\cdot)$ is said to be periodic,
with period θ, if, for all $t \geq 0$,

$$f(t) = f(t + k\theta) \qquad \text{for all integers} \quad k \, .$$

The period of a periodic function is not unique, since if it has period
θ it also has period $2\theta, 3\theta, \ldots$.

One often assumes time to be measured in months. A periodic function
$f(\cdot)$ with period one year then has period $\theta = 12$; that is, for all
$t \geq 0$ and integers $k = 1, 2, \ldots$,

$$f(t) = f(t + 12) = f(t + 24) = \ldots = f(t + 12k) \, .$$

Frequency: A function $f(t)$ is said to be a (real) harmonic with
frequency ω and amplitude A, where ω and A are positive constants,
if it is of the form

$$f(t) = A \cos \omega t$$

or of the form

4

$$f(t) = A \sin \omega t \ .$$

A harmonic with frequency ω has period

$$\theta = \frac{2\pi}{\omega}$$

since for any integer k

$$\cos[\omega(t + \frac{2\pi}{\omega} k)] = \cos(\omega t + 2\pi k) = \cos \omega t$$

$$\sin[\omega(t + \frac{2\pi}{\omega} k)] = \sin(\omega t + 2\pi k) = \sin \omega t \ .$$

If a harmonic has period θ, its frequency ω must satisfy
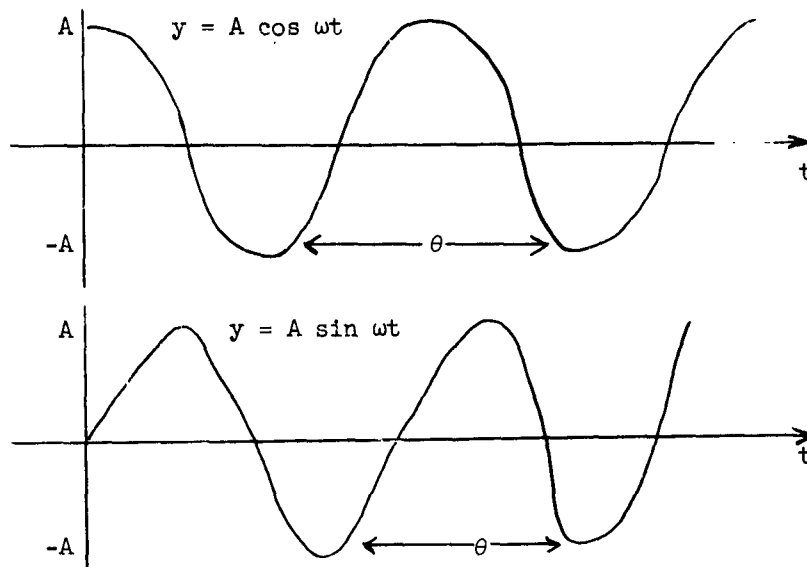
$$\omega = k \ \frac{2\pi}{\theta} \qquad \text{for some integer} \quad k = 1, 2, \ldots$$

since

$$\cos[\omega(t + \theta)] = \cos \omega t \qquad \text{implies} \qquad \theta = k \ \frac{2\pi}{\omega} \ .$$

5

<u>Physical interpretation of frequency</u>. Typical harmonics are sketched

below:

The period θ is given by the distance from peak to peak, or

equivalently it is given by the distance from trough to trough. The

frequency ω represents the number of complete cycles (or swings) in

2π units of time; it is therefore called the angular or radian frequency,

to distinguish it from <u>true</u> <u>frequency</u> ν, <u>measured</u> <u>in</u> <u>cycles</u> <u>per</u> <u>unit</u>

<u>time</u>. The true frequency ν of a harmonic with angular frequency ω

is given by

$$\nu = \frac{\omega}{2\pi} .$$

To illustrate these ideas suppose that time t is measured in

months. The harmonics $\cos \omega t$ and $\sin \omega t$ have period $\frac{2\pi}{\omega}$ and repre-

sent variations which recur every $\frac{2\pi}{\omega}$ months. Thus the harmonics

6

$\cos \frac{2\pi}{6} t$ and $\sin \frac{2\pi}{6} t$ have period 6 and represent values which recur every 6 months. The true frequency of these harmonics is $\frac{1}{6}$ which means that in one unit of time (here, one month) $\frac{1}{6}$ of a swing takes place.

Zero frequency. A constant function

$$f(t) = A$$

can be regarded as the value of the harmonic $A \cos \omega t$ with frequency ω equal to zero. Consequently, by a zero frequency harmonic we mean a constant.

Expansion of a periodic function into harmonics. It is a very remarkable fact that a function $f(t)$ with period θ can be arbitrarily closely approximated by a linear combination

$$(1) \qquad f_N(t) = \sum_{n=0}^{N} \{A_n \cos(n \frac{2\pi}{\theta} t) + B_n \sin(n \frac{2\pi}{\theta} t)\}$$

of the harmonics with frequencies $k\frac{2\pi}{\theta}$, $k = 0, 1, 2, \ldots$. A function of the form of (1) is called a harmonic polynomial of degree N and period θ.

There are many senses in which one can define closeness of approximation. Consequently, in order to form the harmonic polynomial which approximates a periodic function one must first define the mode of approximation.

<u>Least squares approximation over an interval</u>. Let $f(t)$ be a periodic function with period θ. In studying this function we can confine our attention to the interval $0 \leq t \leq \theta$, since whatever happens in this interval is repeated outside of it.

One sense in which a harmonic polynomial $f_N(t)$ of degree N and period θ can be said to be a good approximation to $f(t)$ is if the integrated square difference

$$(2) \qquad \int_0^\theta \{f(t) - f_N(t)\}^2 \, dt$$

is as small as possible; more precisely, the harmonic polynomial $f_N(t)$ defined by (1) is called the least squares approximation to $f(t)$ by a harmonic polynomial of degree N if

$$(3) \quad \int_0^\theta \{f(t) - f_N(t)\}^2 \, dt = \min_{a_0,a_1,b_1,\ldots,a_N,b_N} S(a_0,a_1,b_1,\ldots,a_N,b_N)$$

where

$$
(4) \quad
\begin{aligned}
&S(a_0, a_1, b_1, \ldots, a_N, b_N) \\
&= \int_0^\theta [f(t) - \sum_{n=0}^N \{a_n \cos(n \tfrac{2\pi}{\theta} t) + b_n \sin(n \tfrac{2\pi}{\theta} t)\}]^2 \, dt \ .
\end{aligned}
$$

To determine $A_0, A_1, B_1, \ldots, A_N, B_N$, one finds the roots of the derivatives $\frac{\partial S}{\partial a_m}$ and $\frac{\partial S}{\partial b_m}$ for $m = 0, 1, \ldots, N$. Proceeding in this way one arrives at the following conclusion.

8

Normal equations for the coefficients of the least squares approximating harmonic polynomial. The coefficients $A_0, B_0, A_1, B_1, \ldots, A_N, B_N$ are the solution of the system of linear equations [for $m = 0, 1, \ldots, N$]

$$\sum_{n=0}^{N} \left\{ A_n \int_0^\theta \cos \omega_n t \cos \omega_m t \, dt + B_n \int_0^\theta \sin \omega_n t \cos \omega_m t \, dt \right\}$$

$$= \int_0^\theta f(t) \cos \omega_m t \, dt \, ,$$

(5)

$$\sum_{n=0}^{N} \left\{ A_n \int_0^\theta \cos \omega_n t \sin \omega_m t \, dt + B_n \int_0^\theta \sin \omega_n t \sin \omega_m t \, dt \right\}$$

$$= \int_0^\theta f(t) \sin \omega_m t \, dt \, ,$$

where

$$(6) \qquad \omega_n = n \frac{2\pi}{\theta} \, .$$

The solution of these equations is rendered trivial by the following easily verified facts:

$$(7) \quad
\begin{cases}
\displaystyle\int_0^\theta \cos \omega_n t \cos \omega_m t \, dt = 0 & \text{if } n \neq m \, . \\[2ex]
\displaystyle\int_0^\theta \sin \omega_n t \sin \omega_m t \, dt = 0 & \text{if } n \neq m \, . \\[2ex]
\displaystyle\int_0^\theta \sin \omega_n t \cos \omega_m t \, dt = 0 \, .
\end{cases}$$

$$(8) \begin{cases} \displaystyle\int_0^\theta \cos^2 \omega_n t \, dt = \frac{\theta}{2} & \text{if } n > 0 . \\[4mm] \qquad\qquad\quad = \theta & \text{if } n = 0 . \\[4mm] \displaystyle\int_0^\theta \sin^2 \omega_n t \, dt = \frac{\theta}{2} & \text{if } n > 0 . \\[4mm] \qquad\qquad\quad = 0 & \text{if } n = 0 . \end{cases}$$

Consequently, the least squares approximating harmonic polynomial of degree N to the function $f(t)$ over the interval 0 to θ is given by (1) with

$$(9) \begin{cases} A_0 = \dfrac{1}{\theta} \displaystyle\int_0^\theta f(t) \, dt , \qquad B_0 = 0 . \\[5mm] A_n = \dfrac{2}{\theta} \displaystyle\int_0^\theta f(t) \cos(n \, \tfrac{2\pi}{\theta} t) \, dt , \qquad n > 0 . \\[5mm] B_n = \dfrac{2}{\theta} \displaystyle\int_0^\theta f(t) \sin(n \, \tfrac{2\pi}{\theta} t) \, dt , \qquad n > 0 . \end{cases}$$

It is striking that the expressions in (9) for the coefficients do not depend on N. It may be verified that

$$(10) \begin{aligned} &\int_0^\theta [f(t) - \sum_{n=0}^{N} (A_n \cos \omega_n t + B_n \sin \omega_n t)]^2 \, dt \\ &= \int_0^\theta f^2(t) \, dt - \sum_{n=0}^{N} \{A_n^2 + B_n^2\} . \end{aligned}$$

For any finite N, this integrated squared difference is in general positive. However, it tends to 0 as N tends to ∞; consequently, one may write $f(t)$ as an infinite series of harmonics,

$$(11) \qquad f(t) = \sum_{n=0}^{\infty} \{A_n \cos \omega_n t + B_n \sin \omega_n t\} , \qquad 0 \leq t \leq \theta ,$$

where $\{\omega_n\}$ are given by (6) and $\{A_n\}$ and $\{B_n\}$ are given by (9). The right hand side of (11) is called the <u>Fourier series</u> expansion of $f(t)$. It should be noted that, unless $f(t)$ is a sufficiently well behaved function, this infinite series does not necessarily converge at each value of t. In order to emphasize this fact, instead of writing an equality sign in (11) one sometimes writes

$$(12) \qquad f(t) \sim \sum_{n=0}^{\infty} \{A_n \cos \omega_n t + B_n \sin \omega_n t\}$$

to mean that the expressions in (10) converge to zero as N tends to ∞. However in this paper we do not employ the notation given in (12).

<u>Fourier coefficients</u>: Given any periodic function $f(\cdot)$ one can form sequences $\{A_n\}$ and $\{B_n\}$ defined by (9). The quantities A_n and B_n are called the Fourier coefficients of the function $f(t)$. From (10) it follows that they provide a means of summarizing the continuum of numbers which constitute a function defined over an interval by means of countably infinitely many numbers. Some of the important properties of Fourier coefficients are the following (proofs are left to the reader.)

11

1) If a function $f(t)$ has period θ, its Fourier coefficients can be written in terms of an integral over any complete period t_0 to $t_0 + \theta$:

$$A_0 = \frac{1}{\theta} \int_{t_0}^{t_0+\theta} f(t) \, dt \, ,$$

while for $n \geq t$,

(13)
$$A_n = \frac{2}{\theta} \int_{t_0}^{t_0+\theta} f(t) \, \cos(n \frac{2\pi}{\theta} t) \, dt \, .$$

$$B_n = \frac{2}{\theta} \int_{t_0}^{t_0+\theta} f(t) \, \sin(n \frac{2\pi}{\theta} t) \, dt \, .$$

2) If a periodic function $f(t)$ is even, in the sense that for all t

$$f(-t) = f(t) \, ,$$

then the sine coefficients B_n vanish,

$$B_n = 0 \qquad \text{for all } n \, ,$$

and

(14)
$$A_n = \frac{2}{\theta} \int_0^{\theta/2} f(t) \, dt \, , \qquad n = 0 \, ,$$

$$= \frac{4}{\theta} \int_0^{\theta/2} f(t) \, \cos(n \frac{2\pi}{\theta} t) \, dt \, , \qquad n \geq 1 \, .$$

12

3) If a periodic function $f(t)$ is odd, in the sense that for all t

$$f(-t) = -f(t) \ ,$$

then the cosine coefficients vanish,

$$A_n = 0 \qquad \text{for all } n \ ,$$

and for $n \geq 1$

$$(15) \qquad B_n = \frac{4}{\theta} \int_0^{\theta/2} f(t) \, \sin(n \frac{2\pi}{\theta} t) \, dt \ .$$

4) If the periodic function $f(t)$ is absolutely integrable

$$\int_0^\theta |f(t)| \, dt < \infty \ ,$$

then

$$(16) \qquad \lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = 0 \ .$$

If $f(t)$ is k-times differentiable, and its k-th derivative $f^{(k)}(t)$ is absolutely integrable, then

$$(17) \qquad \lim_{n \to \infty} n^k A_n = \lim_{n \to \infty} n^k B_n = 0 \ .$$

13

<u>Complex Form of the Fourier Coefficients</u>:   The envelope R_n and

phase φ_n of the Fourier coefficients A_n and B_n is defined by (for

$n \geq 0$)

(18) $$R_n = \sqrt{A_n^2 + B_n^2} , \qquad \varphi_n = \tan^{-1} \frac{B_n}{A_n} .$$

It is easily verified that

$$A_n \cos(n \frac{2\pi}{\theta} t) + B_n \sin(n \frac{2\pi}{\theta} t)$$

(19)

$$= R_n \cos(n \frac{2\pi}{\theta} t - \varphi_n) .$$

In terms of the envelope and phase coefficients the Fourier series expan-

sion of a periodic function $f(t)$ may be written

(20) $$f(t) = \sum_{n=0}^{\infty} R_n \cos(n \frac{2\pi}{\theta} t - \varphi_n) .$$

In order to understand the intuitive meaning of the envelope and phase,

we introduce the complex form of the Fourier coefficients.

For ease of manipulation it is frequently desirable to write the

Fourier series as an expansion in complex harmonics (where $i = \sqrt{-1}$):

(21) $$\exp[in \frac{2\pi}{\theta} t] = \cos(n \frac{2\pi}{\theta} t) + i \sin(n \frac{2\pi}{\theta} t) .$$

Thus let C_n be the coefficient in the expansion

$$(22) \qquad f(t) = \sum_{n=-\infty}^{\infty} C_n \exp[in \frac{2\pi}{\theta} t] \; .$$

The quantities C_n are called the complex Fourier coefficients of $f(t)$.

To determine the relation between the various Fourier coefficients, we write (20) in the form

$$f(t) = R_0 + \sum_{n=1}^{\infty} \frac{1}{2} R_n \{\exp[i(n \frac{2\pi}{\theta} t - \varphi_n)] + \exp[i(-n \frac{2\pi}{\theta} t + \varphi_n)]\}$$

$$(23)$$

$$= R_0 + \sum_{n=1}^{\infty} \{\frac{1}{2} R_n \exp(-i\varphi_n)\} \exp(in \frac{2\pi}{\theta} t)$$

$$+ \sum_{n=-1}^{-\infty} \frac{1}{2} R_{-n} \exp(i\varphi_{-n}) \exp(in \frac{2\pi}{\theta} t) \; .$$

Comparing (22) and (23) we see that

$$C_0 = R_0 = A_0 = \frac{1}{\theta} \int_0^{\theta} f(t) \, dt \; ,$$

for $n \geq 1$

$$C_n = \frac{1}{2} R_n \exp(-i\varphi_n)$$

$$= \frac{1}{2} \{R_n \cos \varphi_n - i R_n \sin \varphi_n\}$$

$$= \frac{1}{2} \{A_n - i B_n\}$$

$$= \frac{1}{\theta} \int_0^{\theta} f(t) \{\cos(n \frac{2\pi}{\theta} t) - i \sin(n \frac{2\pi}{\theta} t)\} \, dt$$

$$= \frac{1}{\theta} \int_0^{\theta} f(t) \exp[-i(n \frac{2\pi}{\theta} t)] \, dt \; ,$$

15

and for $n \leq -1$,

$$C_n = \frac{1}{2} R_{-n} \exp(i\varphi_{-n})$$

$$= \frac{1}{2} \{A_{-n} + iB_{-n}\}$$

$$= \frac{1}{\theta} \int_0^\theta f(t) \{\cos(-n \frac{2\pi}{\theta} t)\} dt$$

$$= \frac{1}{\theta} \int_0^\theta f(t) \exp[-i(n \frac{2\pi}{\theta} t)] dt .$$

One sees that the complex Fourier coefficients C_n are given by a single formula (valid for $n = 0, \pm 1, \pm 2, \dots$):

$$(24) \qquad C_n = \frac{1}{\theta} \int_0^\theta f(t) \exp[-i(n \frac{2\pi}{\theta} t)] dt .$$

In general C_n is a complex number, which may be written in polar form:

$$(25) \qquad C_n = R_n e^{i\varphi_n}$$

defining

$$R_{-n} = R_n , \qquad \varphi_{-n} = -\varphi_n .$$

From (25) one sees that the <u>envelope</u> R_n <u>and the phase</u> φ_n <u>are the polar coordinates of the complex Fourier coefficient</u> C_n.

16

Equation (24) is one instance of the fact that the complex Fourier series (23) is easier to manipulate than is the real Fourier series (11). Another instance of this fact is given by the following derivation of the important convolution theorem.

Convolution theorem. Let $f_1(t)$ and $f_2(t)$ be square-integrable functions with period θ, and respective complex Fourier series

$$f_j(t) = \sum_{n=-\infty}^{\infty} (C_j)_n \exp[in \frac{2\pi}{\theta} t] , \qquad j = 1,2 .$$

The function $f_3(\cdot)$ defined by

$$(26) \qquad f_3(t) = \int_0^\theta f_1(s) f_2(t - s) ds$$

is called the convolution of $f_1(\cdot)$ and $f_2(\cdot)$ and is often written

$$f_3 = f_1 * f_2 .$$

Now

$$(27) \qquad f_1(s) f_2(t - s) = \sum_{n=-\infty}^{\infty} f_1(s) (C_2)_n \exp[in \frac{2\pi}{\theta} (t - s)] .$$

Integrating both sides of (27) over the interval $0 \leq s \leq \theta$, it follows that

$$(28) \qquad f_3(t) = \sum_{n=-\infty}^{\infty} \exp[in \frac{2\pi}{\theta} t] (C_1)_n (C_2)_n .$$

17

Equation (28) is a fundamental formula, expressing the convolution of two functions in terms of their Fourier coefficients. From (28) it follows that

$$(29) \quad (C_3)_n = \frac{1}{\theta} \int_0^\theta f_3(t) \exp[-i(n \frac{2\pi}{\theta} t)] \, dt = (C_1)_n (C_2)_n .$$

In words, the complex Fourier coefficient $(C_3)_n$ of the convolution $f_3(\cdot)$ is the product of the complex Fourier coefficients $(C_1)_n$ and $(C_2)_n$.

3. Windows

The notion of a Fourier series derives its usefulness from the idea that it enables one to approximately represent an arbitrary function $f(t)$ as a harmonic polynomial of some finite degree N. The question naturally arises: how large need N be chosen in order that the harmonic polynomial $f_N(t)$ defined by (2.1) can be considered to be a good approximation to $f(t)$? Indeed, the question needs to be examined whether

$$(1) \qquad \lim_{N \to \infty} f_N(t) = f(t)$$

at each t in the interval $0 \le t \le \theta$. In this section we develop the tools to answer such questions.

We first obtain a representation for $f_N(t)$ in terms of $f(t)$. For $n \ge 1$,

$$A_n \cos \omega_n t + B_n \sin \omega_n t$$

$$= \frac{2}{\theta} \int_0^\theta f(x) \{\cos \omega_n x \cos \omega_n t + \sin \omega_n x \sin \omega_n t\} \, dx$$

$$= \frac{2}{\theta} \int_0^\theta f(x) \cos \omega_n (x - t) \, dx \; .$$

Consequently,

$$f_N(\theta) = \frac{1}{\theta} \int_0^\theta \{1 + 2 \cos \omega_1 (x - t) + \ldots + 2 \cos \omega_N (x - t)\} f(x) \, dx$$

$$(2)$$

$$= \int_0^\theta \frac{1}{\theta} D_N(2\pi \frac{x - t}{\theta}) f(x) \, dx$$

19

defining

(3) $\qquad D_N(z) = 1 + 2 \cos z + 2 \cos 2z + \ldots + 2 \cos Nz$.

The function $D_N(z)$ is called Dirichlet's kernel. A graph of $D_N(z)$ is given in Figure 3A.

One may verify that

(4) $$D_N(z) = \frac{\sin(N + \frac{1}{2})z}{\sin \frac{1}{2} z} \quad .$$

To prove (4) we write

$$\sin(\tfrac{1}{2} z) \, D_N(z) = \sin \tfrac{1}{2} z + 2 \sin \tfrac{1}{2} z \cos z + \ldots$$

$$+ 2 \sin \tfrac{1}{2} z \cos Nz$$

$$= \sin \tfrac{1}{2} z + \sin \tfrac{3}{2} z - \sin \tfrac{1}{2} z + \ldots$$

$$+ \sin(N + \tfrac{1}{2})z - \sin(N - \tfrac{1}{2})z$$

$$= \sin(N + \tfrac{1}{2})z \quad .$$

The function $D_N(z)$ is an even periodic function with period 2π. Consequently it suffices to graph it for $|z| \leq \pi$. It achieves its maximum in this interval at $z = 0$, where it has the value

$$D_N(0) = 2N + 1 \ .$$

It has zeros at the points z satisfying $\sin(N + \frac{1}{2})z = 0$ so that $(N + \frac{1}{2})z = \pm\pi, \ \pm2\pi, \ \ldots;$ thus the zeroes of $D_N(z)$ are at

$$z \ = \ \pm \frac{2\pi}{2N + 1} \ , \ \pm \frac{4\pi}{2N + 1} \ , \ \cdots \ .$$

Equation (2) shows that the harmonic polynomial $f_N(t)$ is actually an integral averaging over the values of $f(t)$ weighted by the kernel $\frac{1}{\theta} D_N(2\pi \frac{x-t}{\theta})$. Using an analogy from communication theory, $f_N(t)$ may be regarded as the impression obtained of the function $f(t)$ when it is viewed through a window (or channel) of variable transmission properties given by the weighting function $\frac{1}{\theta} D_N(2\pi \frac{x-t}{\theta})$. We are thus led to the following formal definition of the notion of a window.

If $f(t)$ and $g(t)$ are periodic functions with period θ such that

$$(5) \qquad\qquad g(t) = \int_0^\theta G(t - x) \ f(x) \ dx \ ,$$

we say that $G(t)$ is the window through which $g(t)$ views $f(t)$.

The truncated Fourier series $f_N(t)$ of a function $f(t)$ views the function through a window $\frac{1}{\theta} D_N(\frac{2\pi}{\theta} t)$ which is essentially the Dirichlet kernel. By examining the Dirichlet kernel, we can see some of the properties windows should have.

A window $G(t)$ should be an even function, so that it gives equal treatment to values of $f(\cdot)$ on both sides of a given point t:

$$(6) \qquad\qquad G(-t) = G(t) \ .$$

21

A window G(t) should integrate to 1,

$$(7) \qquad\qquad \int_0^\theta G(t) \, dt = 1 \ ,$$

so that if f(t) is identically a constant c, then its transmitted
value g(t) will be a constant and further the same constant c.

A window G(t) should achieve its maximum at t = 0,

$$(8) \qquad\qquad |G(t)| \le G(0) \qquad \text{for all } \ t \ ,$$

and should be "concentrated" as much as possible about t = 0 in order
that the value g(t) of the transmitted function g(·) at a particular
point t should reflect (as much as possible) the behavior of f(·) in
the neighborhood of t.

The Dirichlet kernel cannot be considered to be satisfactorily con-
centrated about t = 0. One reason for saying this is the relative
(absolute) size of the second largest peak (actually a negative peak, or
trough) to the size of the largest peak. The largest peak of $D_N(z)$,
which occurs at z = 0, is 2N + 1 while the second largest peak
(trough), which occurs approximately (for large N) at z = $\pm 3\pi/(2N + 1)$,
is approximately $-(2N + 1) \, (2/3\pi)$. The ratio of size of second largest
peak to size of largest peak is thus approximately given by (for large N)

$$\frac{2}{3\pi} = 0.21 \ .$$

A second reason why the Dirichlet kernel must be considered an unsatisfactory window comes from the convergence theory of Fourier series. It is not true for an arbitrary continuous function $f(\cdot)$ that the truncated Fourier series $f_N(t)$ converges to $f(t)$ as N tends to ∞; for this to hold, some supplementary condition (such as that the function $f(\cdot)$ be of bounded variation) must be imposed. This fact is somewhat disturbing since it is known (Weierstrass' theorem) that to each continuous function $f(t)$ there exists a sequence of harmonic polynomials $f_N(t)$ satisfying (1).

In order for this existence theorem to be of practical importance, methods must be developed for constructing such approximating harmonic polynomials, preferably using Fourier coefficients. One of the first such methods was discovered by Fejèr. He proved that if one considers the arithmetic mean $\overline{f_N}(t)$ of the first N Fourier sums of a function $f(t)$,

$$(9) \qquad \overline{f_N}(t) = \frac{1}{N+1} \{f_0(t) + f_1(t) + \ldots + f_N(t)\} ,$$

then $\overline{f_N}(t)$ is a harmonic polynomial of degree N, and the sequence $\overline{f_N}(t)$ converges uniformly to $f(t)$ if $f(t)$ is a continuous periodic function.

One can give an explicit expression for $\overline{f_N}(t)$ in terms of the Fourier coefficients of $f(t)$:

$$(10) \qquad \overline{f_N}(t) = \sum_{n=0}^{N} \left(1 - \frac{1}{N+1}\right) \{A_n \cos \omega_n t + B_n \sin \omega_n t\} .$$

As an integral over $f(t)$ one may write

$$(11) \qquad \overline{f_N}(t) = \int_0^\theta \frac{1}{\theta} F_{N+1}(2\pi \frac{x-t}{\theta}) f(x) \, dx \,,$$

defining

$$(12) \qquad F_N(z) = \frac{1}{N+1} \left\{ \frac{\sin(N+1)\frac{1}{2}z}{\sin \frac{1}{2}z} \right\}^2 \,.$$

We call $F_N(z)$ the Fejèr kernel. It is graphed in Figure 3A. In words, we may express (11) as follows: the N-th Fejèr approximation $\overline{f_N}(t)$ corresponds to viewing $f(t)$ through the window $\frac{1}{\theta} F_N(2\pi \frac{x-t}{\theta})$.

To prove (11), one uses (2), and the trigonometric identity

$$(13) \qquad \sum_{n=0}^N \frac{\sin(n+\frac{1}{2})z}{\sin \frac{1}{2}z} = \left\{ \frac{\sin(N+1)\frac{1}{2}z}{\sin \frac{1}{2}z} \right\}^2 \,.$$

The Fejèr kernel seems to provide a more efficient window than the Dirichlet kernel since it is more "concentrated" about $t = 0$. The largest peak of $F_N(z)$ is $N+1$ while the second largest peak [at approximately $z = 3\pi/(N+1)$ for large N] is approximately $(N+1)(\frac{2}{3\pi})^2$. The ratio of second largest peak to the largest peak is thus (for large N)

$$\left(\frac{2}{3\pi}\right)^2 = 0.04 \,,$$

compared to 0.21 in the case of the Dirichlet kernel.

It should be pointed out that the ratio of size of second largest peak to size of largest peak is only one of the ways that have been suggested to compare (in a very rough way) the degree of "concentration"

24

of two windows; however it would seem to be valid only in the case that their second largest peaks occur at the same point. Unfortunately there does not seem to be any universally valid criterion for comparing the "concentration" of two windows; in the sequel, a number of such criteria will be mentioned.

The introduction of the Fejèr means $\overline{f}_N(t)$ makes us aware that the truncated Fourier series is not the only way in which the first N Fourier coefficients of a function may be used to form a harmonic approximation. Various other possibilities that have been suggested are as follows.

The Modified Truncated Fourier Series. Because of the window to which it leads, various authors have considered instead of the truncated Fourier series $f_N(t)$ the modified expression

$$(14) \qquad f_N^*(t) = f_N(t) - \frac{1}{2} \{A_N \cos \omega_n t + B_N \sin \omega_n t\} .$$

It is easily verified that

$$(15) \qquad f_N^*(t) = \int_0^\theta \frac{1}{\theta} D_N^*(2\pi \frac{x - t}{\theta}) f(x) \, dx$$

where $D_N^*(z)$ is the modified Dirichlet kernel defined by

$$D_N^*(z) = D_N(z) - \cos Nz$$

$$= 1 + 2 \cos z + 2 \cos 2z + \ldots + 2 \cos(N - 1)z + \cos Nz$$

$$= \frac{\sin Nz}{\tan \frac{1}{2} z}$$

(see Figure 3A for a graph of $D_N^*(z)$.)

25

Tukey means. Another way to form a harmonic approximation to $f(t)$
is to form the mean

(17) $\qquad f_N^{(T)}(t) = \frac{1}{2} f_N^*(t) + \frac{1}{4} f_N^*(t + \frac{\theta}{2N}) + \frac{1}{4} f_N^*(t - \frac{\theta}{2N})$

which the author calls the Tukey mean (of the truncated Fourier series)
because of its relation to certain procedures recommended by John W. Tukey
(see Blachman and Tukey (1958), p. 14) for the spectral analysis of time
series. To find the window corresponding to Tukey means, we write

$$f_N^*(t) = \int_0^\theta \frac{1}{\theta} \left\{ \frac{1}{2} D_N^*\!\left(\frac{2\pi}{\theta} [x - t]\right) + \frac{1}{4} D_N^*\!\left(\frac{2\pi}{\theta} [x - t] - \frac{\pi}{N}\right) \right.$$

(18) $$\left. + \frac{1}{4} D_N^*\!\left(\frac{2\pi}{\theta} [x - t] + \frac{\pi}{N}\right) \right\} f(x) \; dx$$

$$= \int_0^\theta \frac{1}{\theta} T_N\!\left(\frac{2\pi}{\theta} [x - t]\right) f(x) \; dx$$

defining

(19) $\qquad T_N(z) = \frac{1}{2} D_N^*(z) + \frac{1}{4} D_N^*\!\left(z + \frac{\pi}{N}\right) + \frac{1}{4} D_N^*\!\left(z - \frac{\pi}{N}\right) .$

We call $T_N(z)$ the Tukey kernel (it is graphed in Figure 3A). Intuitively,
one feels that the Tukey kernel is more "concentrated" than either the
Dirichlet kernel or the Fejèr kernel, and therefore provides a better
harmonic approximation.

It is possible to explain how one might be led to discover the Tukey
kernel. The unsatisfactory feature of the modified Dirichlet kernel (like

26

the Dirichlet kernel) is that the absolute size of its second largest peak (actually, trough), which occurs at $z = 3\pi/2N$, relative to its largest peak at $z = 0$, is approximately $1/5$. To reduce the size of the secondary peak, one combines $D_N^*(z)$ with a shifted version of itself, namely $D_N^*(z + \frac{\pi}{N})$, which has enough positivity at $z = 3\pi/2N$ to reduce the negative trough which $D_N^*(z)$ possesses there. It is thus possible to arrive at a class of possible kernels,

$$(20) \qquad T_N^{(\alpha)}(z) = \alpha\, D_N^*(z) + \frac{1}{2}\,(1 - \alpha)\left\{D_N^*\left(z + \frac{\pi}{N}\right) + D_N^*\left(z - \frac{\pi}{N}\right)\right\}$$

where α is chosen in $0 < \alpha < 1$. R. W. Hamming and J. W. Tukey, at the Bell Telephone Laboratories, experimented with various choices of α, and at one time recommended $\alpha = 0.54$.

Lanczos means. The mean

$$(21) \qquad f_N^{(C)}(t) = \frac{4N}{\theta} \int_{t-\frac{\theta}{2N}}^{t+\frac{\theta}{2N}} f_N(t')\, dt'$$

has essentially been recommended by C. Lanczos (see Lanczos (1956), p. 227). It may be verified that

$$(22) \qquad f_N^{(C)}(t) = \frac{1}{\theta} \int_0^{\theta} L_N\left(\frac{2\pi}{\theta}\,[x - t]\right) f(x)\, dx\ ,$$

defining the Lanczos kernel

$$(23) \qquad L_N(z) = \frac{2N}{\pi} \int_{-\pi/N}^{\pi/N} D_N(z + u)\, du\ .$$

Like the Tukey kernel, the Lanczos kernel achieves greater concentration about $z = 0$ by averaging over the Dirichlet kernel so as to reduce the secondary peaks.

4. Coefficient Windows

One way of describing how well a harmonic polynomial approximates a function $f(t)$ is by specifying the window through which the polynomial views $f(t)$. From the point of view of constructing approximating harmonic polynomials, it is preferable to use formulas for the polynomials in terms of the Fourier coefficients of the function $f(t)$. We are thus led to the notion of a coefficient window.

The harmonic approximation to a periodic function $f(t)$ with period θ with coefficient window $\{k_N(n),\ n = 0, 1, \ldots , N\}$ is defined to be the harmonic polynomial

$$(1) \qquad f_{N;k_N}(t) = \sum_{n=0}^{N} k_N(n) \{A_n \cos \omega_n t + B_n \sin \omega_n t\}$$

where $\omega_n = n(2\pi/\theta)$ and A_n and B_n are the Fourier coefficients of $f(t)$.

In Table 4A we tabulate the coefficient windows corresponding to the various harmonic approximations considered in Section 3.

The Fourier transform

$$(2) \qquad K_N(z) = k_N(0) + 2 \sum_{j=1}^{N} \cos jz\ k_N(j)$$

is defined to be the amplitude window of the harmonic approximation. Using (1) one may verify that

$$(3) \qquad f_{N;k_N}(t) = \int_0^\theta \frac{1}{\theta} K_N\left(\frac{2\pi}{\theta} [x - t]\right) f(x)\ dx .$$

29

From (3) one sees that the true window through which the harmonic approximation $f_{N;k_N}(t)$ views $f(x)$ is [up to a change of scale depending on the period θ of $f(t)$] the amplitude window $K_N(z)$. Note that $K_N(z)$ has period 2π.

At $n = 0$, the coefficient window always has the value 1,

(4) $$k_N(0) = 1 ,$$

since we desire that

$$\frac{1}{2\pi} \int_0^{2\pi} K_N(z) \, dz = 1 .$$

We also define the coefficient window for negative values of n in such a way that it is an even function:

(5) $$k_N(-n) = k_N(n) , \qquad n = 0, \pm 1, \ldots , \pm N .$$

Instead of (2) we may then write

(6) $$K_N(z) = \sum_{j=-N}^{N} \cos jz \; k_N(j) .$$

Inverting (6) we have

(7) $$k_N(n) = \frac{1}{2\pi} \int_0^{2\pi} \cos nz \; K_N(z) \, dz .$$

One advantage of introducing the coefficient windows and amplitude windows is that the period θ of the periodic function $f(\cdot)$ under

consideration no longer explicitly enters. However, a more important advantage is that we can give a general treatment of the properties of harmonic approximations. In order to do this it is convenient to introduce still another stage of abstraction.

If $k(u)$ is a function, defined for $-\infty < u < \infty$, such that

i) $k(0) = 1$

ii) $k(-u) = k(u)$

iii) $k(u) = 0$ for $|u| > 1$

we call $k(u)$ a _coefficient window generator_. Its Fourier transform

$$(8) \qquad K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} k(u) \, du \, , \qquad -\infty < z < \infty \, ,$$

is called an _amplitude window generator_. Note that

$$(9) \qquad k(u) = \int_{-\infty}^{\infty} e^{iuz} K(z) \, dz \; ;$$

since $K(z)$ is not necessarily absolutely integrable over $-\infty < z < \infty$, the integral in (9) is in general defined as a limit in mean square.

Now, let M be a positive number (in practice, M will be a member of a sequence M_N of numbers which tend to ∞ as N tends to ∞). One may define a coefficient window $k_N(u)$ by

$$(10) \qquad k_N(n) = k\left(\frac{n}{M}\right) \, .$$

Note that the window defined by (10) satisfies (4) and (5).

31

The coefficient windows in Table 4A can be generated by a formula of the form of (10) from a coefficient window generator (see Table 4B).

The relationship that exists between M and N is determined by the requirement that

$$(11) \qquad k\left(\frac{n}{M}\right) = 0 \qquad \text{for} \quad n = N + 1, N + 2, \ldots ;$$

this assumption holds in all the cases considered in Table 4B.

We now obtain a formula for the amplitude window corresponding to the coefficient window defined by (10). Using (11), the harmonic polynomial (1) may then be written

$$f_{N;k_N}(t) = \sum_{n=0}^{N} k\left(\frac{n}{M}\right) \{A_n \cos \omega_n t + B_n \sin \omega_n t\}$$

$$(12)$$

$$= \sum_{n=0}^{\infty} k\left(\frac{n}{M}\right) \{A_n \cos \omega_n t + B_n \sin \omega_n t\} .$$

We next use (9) to obtain

$$f_{N;k_N}(t) = \sum_{n=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \cos\left(\frac{nz}{M}\right) K(z) \, dz \right\} \{A_n \cos \omega_n t + B_n \sin \omega_n t\}$$

$$(13) \qquad = \int_{-\infty}^{\infty} dz \, K(z) \sum_{n=0}^{\infty} \left\{ A_n \cos\left(n \frac{2\pi t}{\theta}\right) \cos\left(n \frac{z}{M}\right) \right.$$

$$\left. + B_n \sin\left(n \frac{2\pi t}{\theta}\right) \cos\left(n \frac{z}{M}\right) \right\} .$$

Since

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \sin(A + B) + \sin(A - B)$$

it follows that the last sum in (13) is equal to

$$\frac{1}{2} \sum_{n=0}^{\infty} \left\{ A_n \left[\cos \omega_n \left(t + \frac{\theta}{2\pi} \frac{z}{M} \right) + \cos \omega_n \left(t - \frac{\theta}{2\pi} \frac{z}{M} \right) \right] \right.$$

$$\left. + B_n \left[\sin \omega_n \left(t + \frac{\theta}{2\pi} \frac{z}{M} \right) + \sin \omega_n \left(t - \frac{\theta}{2\pi} \frac{z}{M} \right) \right] \right\}$$

$$= \frac{1}{2} \left\{ f\left(t + \frac{\theta}{2\pi} \frac{z}{M} \right) + f\left(t - \frac{\theta}{2\pi} \frac{z}{M} \right) \right\} .$$

Consequently, recalling that $f(t)$ is a periodic function defined for all t in $-\infty < t < \infty$,

(14)
$$f_{N;k_N}(t) = \frac{1}{2} \int_{-\infty}^{\infty} dz \, K(z) \left\{ f\left(t + \frac{\theta}{2\pi} \frac{z}{M} \right) + f\left(t - \frac{\theta}{2\pi} \frac{z}{M} \right) \right\}$$

$$= \int_{-\infty}^{\infty} dz \, K(z) \, f\left(t + \frac{\theta}{2\pi} \frac{z}{M} \right) ;$$

to prove the last equation note that $K(z)$ is an even function so that letting $z' = -z$,

$$\int_{-\infty}^{\infty} dz \, K(z) \, f\left(t - \frac{\theta}{2\pi} \frac{z}{M} \right) = \int_{-\infty}^{\infty} dz' \, K(z') \, f\left(t + \frac{\theta}{2\pi} \frac{z'}{M} \right) .$$

In (14) we make the change of variable $y = t + (\theta z / 2\pi M)$, and obtain the following conclusion; for the coefficient window defined by (10),

33

$$f_{N; k_N}(t) = \int_{-\infty}^{\infty} \frac{2\pi M}{\theta} \; K\left(\frac{2\pi M}{\theta}(t - y)\right) f(y) \, dy \; .$$

In words, this conclusion may be expressed as follows: the harmonic approximation $f_{N; k_N}(t)$ corresponds to viewing $f(t)$ through a window on the infinite real line with transmission function

$$(15) \qquad\qquad \frac{2\pi M}{\theta} \; K\left(\frac{2\pi M}{\theta} t\right)$$

which is (up to a scale factor) the amplitude window generator $K(z)$ corresponding to the coefficient window generator $k(u)$. By writing the window relating $f_{N; k_N}(t)$ and $f(t)$ as a function on the infinite real line we have reduced the study of its properties to the study of the properties of the amplitude window generator $K(z)$.

It is of interest to obtain in terms of $K(z)$ an expression for the window relating $f_{N; k_N}(t)$ and $f(t)$ as a function on the basic interval $-\theta/2 \leq t \leq \theta/2$. From (14) one may write

$$(16) \qquad f_{N; k_N}(t) = \sum_{j=-\infty}^{\infty} \int_{(2k-1)\frac{\theta}{2}}^{(2k+1)\frac{\theta}{2}} \frac{2\pi M}{\theta} \; K\left(\frac{2\pi M}{\theta}(t - y)\right) f(y) \, dy \; .$$

In each integral in (16) make the change of variable $z = y - j\theta$. Since

$$f(z + j\theta) = f(z)$$

we obtain

$$(17) \qquad f_{N; k_N}(t) = \int_{-\theta/2}^{\theta/2} dz \, f(z) \sum_{j=-\infty}^{\infty} \frac{2\pi M}{\theta} \; K\left(\frac{2\pi M}{\theta}(t - z - j\theta)\right) \; .$$

34

Thus the window relating $f_{N;k_N}(t)$ and $f(t)$ on the interval $-\theta/2 \leq t \leq \theta/2$ is

$$(18) \qquad \frac{2\pi M}{\theta} \sum_{j=-\infty}^{\infty} K\left(M \frac{2\pi}{\theta} t - M2\pi j\right) .$$

This fundamental result may be summed up as follows:

The amplitude window

$$(19) \qquad K_M(z) = \sum_{n=-\infty}^{\infty} \cos nz \; k\left(\frac{n}{M}\right), \qquad 0 \leq z \leq 2\pi$$

corresponding to the coefficient window $k\left(\frac{n}{M}\right)$ is given by

$$(20) \qquad K_M(z) = 2\pi M \sum_{j=-\infty}^{\infty} K\Big(M(z - 2\pi j)\Big)$$

where $K(z)$ is the amplitude window generator obtained from the coefficient window generator $k(u)$ by (8).

To illustrate the meaning of the fundamental relation given by (20), let us consider the example of Fejèr means. If $f(t)$ is a function with period θ and with Fourier coefficients A_n and B_n, and if

$$(21) \qquad f_{N,F}(t) = \sum_{n=0}^{N} \left(1 - \frac{|n|}{N}\right) \{A_n \cos \omega_n t + B_n \sin \omega_n t\} ,$$

then by Table 4A

$$(22) \qquad f_{N,F}(t) = \int_0^\theta \frac{1}{\theta} K_N\left(\frac{2\pi}{\theta}(x - t)\right) f(x) \; dx$$

where

$$(23) \qquad K_N(z) = \frac{1}{N} \left\{ \frac{\sin \frac{1}{2} Nz}{\sin \frac{1}{2} z} \right\}^2 .$$

However, the harmonic polynomial in (21) can be written

$$(24) \qquad f_{N,F}(t) = \sum_{n=0}^{N} k\left(\frac{u}{M}\right) \{A_n \cos \omega_n t + B_n \sin \omega_n t\}$$

where $M = N$ and

$$k(u) = 1 - |u| , \qquad |u| \leq 1$$
$$(25)$$
$$= 0 , \qquad |u| > 1$$

with Fourier transform

$$K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} k(u) \, du = \frac{1}{\pi} \int_{0}^{1} (1 - u) \cos uz \, du$$
$$(26)$$
$$= \frac{1}{2\pi} \left\{ \frac{\sin(z/2)}{z/2} \right\}^2 .$$

What (20) says is that

$$(27) \qquad \frac{1}{N} \left\{ \frac{\sin \frac{1}{2} Nz}{\sin \frac{1}{2} z} \right\}^2 = N \sum_{j=-\infty}^{\infty} \left\{ \frac{\sin N(z - 2\pi j)/2}{N(z - 2\pi j)/2} \right\}^2 .$$

An independent verification that (27) holds can be obtained by using the expansion

(28)
$$\frac{1}{\sin^2 z} = \sum_{j=-\infty}^{\infty} \frac{1}{(z + \pi j)^2} \ .$$

To do this note that (27) holds if

$$\left\{\frac{\sin \frac{1}{2} Nz}{\sin \frac{1}{2} z}\right\}^2 = \sum_{j=-\infty}^{\infty} \left\{\frac{\sin N(z - 2\pi j)/2}{\frac{1}{2}(z - 2\pi j)}\right\}^2$$

$$= \sum_{j=-\infty}^{\infty} \left\{\frac{\sin \frac{1}{2} Nz}{\frac{1}{2} z - \pi j}\right\}^2$$

which is an immediate consequence of (28).

We next use (20) and (28) to derive line VI in Table 4A: if

$$k(u) = 1 - 6u^2 + 6|u|^3 , \qquad |u| \le \frac{1}{2} ,$$

(29)
$$= 2(1 - |u|)^3 , \qquad \frac{1}{2} < |u| < 1 ,$$

$$= 0 , \qquad |u| > 1 ,$$

if

(30)
$$K_M(z) = \sum_{n=-N}^{N} k\!\left(\frac{n}{M}\right) \cos nz ,$$

and if M is an even integer less than N, then

(31)
$$K_M(z) = \frac{3}{4M^3} \left\{\frac{\sin \frac{1}{4} Mz}{\frac{1}{2} \sin \frac{z}{2}}\right\}^4 \left\{1 - \frac{2}{3}\left(\sin \frac{z}{2}\right)^2\right\} \ .$$

To prove (31) we use the fact, which follows from (20), that

37

(32)
$$K_M(z) = 2\pi M \sum_{j=-\infty}^{\infty} K(M(z - 2\pi j))$$

where

(33)
$$K(z) = \frac{3}{4\pi} \left\{ \frac{\sin z/4}{z/4} \right\}^4 .$$

Therefore, assuming M is even,

$$K_M(z) = \frac{3}{4} M \sum_{j=-\infty}^{\infty} \left\{ \sin\left(\frac{Mz}{4} - \frac{M2\pi j}{4}\right) \right\}^4 \Big/ \left\{ \frac{M}{4}\left(z - 2\pi j\right) \right\}^4$$

(34)
$$= \frac{3}{4M^3} \sum_{j=-\infty}^{\infty} \left\{ \sin \frac{Mz}{4} \right\}^4 \Big/ \left\{ \frac{1}{2}\left(\frac{z}{2} - \pi j\right) \right\}^4 ,$$

$$= \frac{3}{4M^3} \left\{ 2 \sin \frac{Mz}{4} \right\}^4 2^4 \sum_{j=-\infty}^{\infty} \left(\frac{z}{2} - \pi j\right)^{-4} .$$

To sum the last series in (34) we differentiate (28) and obtain:

$$\frac{d}{dz} \sum_{j=-\infty}^{\infty} (z + \pi j)^{-2} = (-2) \sum_{j=-\infty}^{\infty} (z + \pi j)^{-3}$$

$$= \frac{d}{dz} \sin^{-2} z = (-2) \sin^{-3} z \cos z ,$$

$$\frac{d}{dz} \sum_{j=-\infty}^{\infty} (z + \pi j)^{-3} = (-3) \sum_{j=-\infty}^{\infty} (z + \pi j)^{-4}$$

$$= \frac{d}{dz} \{ \sin^{-3} z \cos z \}$$

$$= -\sin^{-3} z \sin z + \cos z (-3) \sin^{-4} z \cos z$$

$$= (-3) \sin^{-4} z \left\{ \cos^2 z + \frac{1}{3} \sin^2 z \right\}$$

$$= (-3) \sin^{-4} z \left\{ 1 - \frac{2}{3} \sin^2 z \right\} .$$

Thus

$$(35) \qquad \sum_{j=-\infty}^{\infty} (z + \pi j)^{-4} = \sin^{-4} z \left\{ 1 - \frac{2}{3} \sin^2 z \right\} ,$$

$$(36) \qquad \sum_{j=-\infty}^{\infty} \left(\frac{z}{2} - \pi j \right)^{-4} = \sin^{-4} \frac{z}{2} \left\{ 1 - \frac{2}{3} \sin^2 \frac{z}{2} \right\} .$$

Combining (36) and (34), one obtains (31).

From (20) we also obtain an <u>approximate</u> expression for the amplitude window

$$(37) \qquad K_M(z) = \sum_{n=-\infty}^{\infty} \cos nz \; k\!\left(\frac{n}{M} \right) , \qquad 0 \le z \le 2\pi$$

corresponding to the coefficient window $k\!\left(\dfrac{u}{M} \right)$: approximately,

$$(38) \qquad K_M(z) = 2\pi M \, K(Mz) .$$

In writing (38), we are assuming that $K(z)$ becomes negligibly small for $|z| \ge 2\pi M$.

5. Evaluation of Fourier coefficients using equi-spaced points.

It may be difficult to obtain the Fourier coefficients of a periodic function $f(t)$, with period θ, because of the difficulty in evaluating the integrals defining the coefficients. Further, the function $f(t)$ may not be known at all points t in the interval $0 \leq t \leq \theta$, but may be known (especially if it is an empirically observed function) only at a set of equi-distantly spaced points. In this section we consider the problem of evaluating the Fourier coefficients

$$A_0 = \frac{1}{\theta} \int_0^\theta f(t) \, dt \, ,$$

(1)
$$A_n = \frac{2}{\theta} \int_0^\theta f(t) \, \cos\left(n \, \frac{2\pi}{\theta} t\right) dt \, , \qquad n \geq 1 \, ,$$

$$B_n = \frac{2}{\theta} \int_0^\theta f(t) \, \sin\left(n \, \frac{2\pi}{t} t\right) dt \, , \qquad n \geq 1 \, ,$$

when the values of the function $f(t)$ are known only at the r equispaced points $t_0, t_1, \cdots, t_{r-1}$ defined by

(2)
$$t_j = \frac{\theta}{r} j \, , \qquad j = 0, 1, \cdots, r - 1 \, .$$

In the theory of Riemann integration it is shown that if $g(t)$ is a continuous function, then

(3)
$$\int_0^\theta g(t) \, dt = \lim_{r \to \infty} \frac{\theta}{r} \sum_{j=0}^{r-1} g\left(\frac{\theta}{r} j\right) \, .$$

In view of (3), it seems natural to approximate the Fourier coefficients

A_0, A_n, and B_n by the respective quantities

$$A_{0,r} = \frac{1}{r} \sum_{j=0}^{r-1} f(t_j)$$

(4)
$$A_{n;r} = \frac{2}{r} \sum_{j=0}^{r-1} f(t_j) \cos\left(n \frac{2\pi}{r} j\right)$$

$$B_{n;r} = \frac{2}{r} \sum_{j=0}^{r-1} f(t_j) \sin\left(n \frac{2\pi}{r} j\right).$$

The quantities defined in (4) are called the Fourier coefficients of $f(t)$ over the discrete set of points $\{t_0, t_1, \ldots, t_r\}$. For brevity we sometimes call them the discrete Fourier coefficients of $f(t)$.

Fortunately it is possible to give exact expressions for the relations between the Fourier coefficients and the discrete Fourier coefficients. We first notice that the set of numbers $\{A_{n;r}, n = 0, 1, 2, \ldots\}$ actually consists of at most r different integers! Indeed if n is an integer from 1 to r, then for any integer k it holds for any t in $\{t_0, t_1, \ldots, t_r\}$

$$\cos(kr + n) \frac{2\pi}{\theta} t_j = \cos\left(n \frac{2\pi}{\theta} t_j\right),$$

(5)
$$\cos(kr - n) \frac{2\pi}{\theta} t_j = \cos\left(n \frac{2\pi}{\theta} t_j\right).$$

In words, given values only at the points $t_0, t_1, \ldots, t_r$, a harmonic $\cos m \frac{2\pi}{\theta} t$, with frequency $m \frac{2\pi}{\theta}$, cannot be distinguished from a harmonic $\cos n \frac{2\pi}{\theta} t$ if m is related to n by

(6)
$$m = kr + n \qquad \text{or} \qquad m = kr - n$$

41

for some integer k. To describe this situation we introduce the following intuitive terminology: we call the frequency $m\frac{2\pi}{\theta}$ an **alias** of the frequency $n\frac{2\pi}{\theta}$ over the points $t_j = j\frac{\theta}{r}$ if m is related to n by (6). The fact that aliases exist is called the "aliasing effect of sampling."

For sines, instead of (5) we have

(7)

$$\sin(kr + n)\frac{2\pi}{\theta}t_j = \sin\left(n\frac{2\pi}{\theta}t_j\right)$$

$$-\sin(kr - n)\frac{2\pi}{\theta}t_j = \sin\left(n\frac{2\pi}{\theta}t_j\right).$$

The proof of (5) and (7) is immediate; for $j = 0, 1, \ldots, r$,

$$\cos\left\{(kr + n)\frac{2\pi}{\theta}\left(\frac{\theta}{r}j\right)\right\} = \cos\left\{(kr + n)\frac{2\pi j}{r}\right\}$$

$$= \cos\left(kr\frac{2\pi j}{r}\right)\cos\left(n\frac{2\pi j}{r}\right) - \sin\left(kr\frac{2\pi j}{r}\right)\sin\left(n\frac{2\pi j}{r}\right)$$

$$= \cos\left(n\frac{2\pi j}{r}\right) = \cos\left\{n\frac{2\pi}{\theta}\left(\frac{\theta}{r}j\right)\right\};$$

$$\sin\left\{(kr + n)\frac{2\pi}{\theta}\left(\frac{\theta}{r}j\right)\right\} = \sin\left\{(kr + n)\frac{2\pi j}{r}\right\}$$

$$= \sin\left(kr\frac{2\pi j}{r}\right)\cos\left(n\frac{2\pi j}{r}\right) + \cos\left(kr\frac{2\pi j}{r}\right)\sin\left(n\frac{2\pi j}{r}\right)$$

$$= \sin\left(n\frac{2\pi j}{r}\right) = \sin\left\{n\frac{2\pi}{\theta}\left(\frac{\theta}{r}j\right)\right\};$$

and so on.

Pictorial representation of aliasing. Let us represent the frequencies $n \frac{2\pi}{\theta}$ on a line running from 0 to ∞, with the integer n on the line representing the frequency $n \frac{2\pi}{\theta}$. Then we can represent the aliasing as a "folding" of the line into the interval 0 to r (see Figure 5A). There is thus a confusion of frequencies due to the fact that two harmonics whose frequencies are related by (6) have the same values at the sampling points t_j. One refers to this confusion of frequencies as "aliasing" because one can regard the various different frequencies $(kr + n) \frac{2\pi}{\theta}$ and $(kr - n) \frac{2\pi}{\theta}$ as adopting the name of the frequency $n \frac{2\pi}{\theta}$ which is the lowest positive frequency of all those which are aliased.

The number of distinct Fourier coefficients. From (5) and (7) one sees that in order to know the values of all the discrete Fourier coefficients $A_{n;r}$ and $B_{n;r}$ it suffices to know them for

$$n = 0, 1, \ldots , \frac{1}{2}(r - 1) \qquad \text{if } r \text{ is odd},$$

(8)

$$= 0, 1, \ldots , \frac{1}{2} r \qquad \text{if } r \text{ is even}.$$

The two equations in (8) can be expressed by one equation which holds for any value of r:

(8')
$$n = 0, 1, \ldots , \left[\frac{1}{2} r \right]$$

where the square brackets indicate that we take the integral part of the number within the brackets.

Now $B_0 = 0$ and if r is even then $B_{r/2} = 0$. Consequently the r observation points $f(t_j)$, $j = 0, 1, \ldots, r - 1$ always yield exactly r distinct discrete Fourier coefficients. Further, the relation between these discrete Fourier coefficients $A_{n;r}$ and $B_{n;r}$ is given by

$$A_{0;r} = A_0 + \sum_{k=1}^{\infty} A_{rk}$$

(9)
$$A_{n;r} = A_n + \sum_{k=1}^{\infty} (A_{rk+n} + A_{rk-n})$$

$$B_{n;r} = B_n + \sum_{k=1}^{\infty} (B_{rk+n} - B_{rk-n}) .$$

To prove (9), substitute the Fourier series expansion

$$f(t) = \sum_{n=0}^{\infty} \{A_n \cos n \frac{2\pi}{\theta} t + B_n \sin n \frac{2\pi}{\theta} t\}$$

in (4), and use (5) and (7) as well as the fact that

$$\left. \begin{array}{c} \sum_{j=0}^{n-1} \cos\left(n \frac{2\pi}{r} j\right) \\[2em] \sum_{j=0}^{r-1} \sin\left(n \frac{2\pi}{r} j\right) \end{array} \right\} = 0 \qquad \text{if} \qquad n \neq 0, r, 2r, \ldots$$

which follows from the fact that

$$\sum_{j=0}^{r-1} \exp\left[i n \frac{2\pi}{r} j\right] = \frac{1 - \exp[i n 2\pi]}{1 - \exp\left[i n \frac{2\pi}{r}\right]} .$$

From (9) one obtains the following conclusion: given the values of a function at a discrete set of points, one can not obtain the true harmonic spectrum of the function which is contained in the Fourier coefficients. Rather one can obtain estimates of the Fourier coefficients of the low frequency harmonics, these estimates being somewhat "contaminated" by higher frequency contributions. In order for the discrete Fourier coefficients $A_{n;r}$ and $B_{n;r}$ to be satisfactory approximations to the Fourier coefficients A_n and B_n, it is necessary and sufficient that the Fourier coefficients be negligible for

$$n > \frac{1}{2} (r - 1) \qquad \text{if } r \text{ is odd },$$

$$n > \frac{1}{2} r \qquad \text{if } r \text{ is even }.$$

For example, if the Fourier coefficients are of the order of $1/n^2$, then the difference between $A_{n;r}$ and A_n (and similarly between $B_{n;r}$ and B_n) is of the order of $1/r^2$.

Nevertheless, if our aim is merely to approximate the function $f(t)$ by a harmonic polynomial, knowing the values of the function at a discrete set of points provides almost as good an approximation as knowing it over an interval.

Harmonic interpolation: Let $f(t)$ be a function with period θ, and discrete Fourier coefficients $A_{n;r}$ and $B_{n;r}$ defined by (4). Given a coefficient window $k_N(n)$, the function

$$(10) \quad f_{I;k_N}(t) = \sum_{n=0}^{N} k_N(n) \left\{ A_{n;r} \cos\left(n \ \frac{2\pi}{\theta} \ t \right) + B_{n;r} \sin\left(n \ \frac{2\pi}{\theta} \ t \right) \right\}$$

45

defines a harmonic approximation to $f(t)$ which can be regarded as arising from interpolation of the values of the function at $t_0, \ldots, t_{r-1}$; the subscript I is used to connote interpolation. It is shown below that in terms of the values of $f(t)$ at these points,

$$(11) \qquad f_{I;k_N}(t) = \sum_{j=0}^{r-1} f(t_j) \frac{1}{r} K_N\left(\frac{2\pi}{\theta}(t - t_j)\right)$$

where $K_N(z)$ is the amplitude window corresponding to the coefficient window $k_N(n)$. It should be noted that the right hand side of (11) can be regarded as a discrete approximation to the Riemann integral on the right hand side of equation (3) of section 4.

To prove (11), we write for $n \geq 1$ (defining $\omega_n = 2\pi n/\theta$)

$$A_{n;r} \cos \omega_n t + B_{n;r} \sin \omega_n t$$

$$= \frac{2}{r} \sum_{j=0}^{r-1} f(t_j) \{\cos \omega_n t_j \cos \omega_n t + \sin \omega_n t_j \sin \omega_n t\}$$

$$= \frac{2}{r} \sum_{j=0}^{r-1} f(t_j) \cos \omega_n (t - t_j) .$$

Consequently

$$f_{I;k_N}(t) = \frac{1}{r} \sum_{j=0}^{r-1} f(t_j) \{1 + 2k_N(1) \cos \omega_1 (t - t_j) + \ldots$$

$$+ 2k_N(N) \cos \omega_n (t - t_j)\} .$$

The proof of (11) is complete.

46

From (11) one sees that for certain choices of $k_N(n)$ the inter-polatory polynomial is _exactly equal_ to the function $f(t)$ at the points t_j; in symbols,

(12) $\qquad f_{I;k_N}(t_j) = f(t_j) \qquad$ for $\qquad j = 0, 1, \ldots, r$.

In order for (12) to hold it suffices that

$$\frac{1}{r} K_N(0) = 1 ,$$

(13)

$$K_N\!\left(\frac{2\pi}{r} j\right) = 0 \qquad \text{for any integer} \qquad j \neq 0 .$$

For a given value of r, (13) implies the following relation between r and N for the various coefficient windows we have considered:

Dirichlet	$r = 2N + 1$
Modified Dirichlet	$r = 2N$
Fejèr	$r = N + 1$
Tukey	$r = N$
Parzen	$r = N/2$

Explicitly, we have the following relations for $t = \{t_0, \ldots, t_{r-1}\}$, defining for any integers n and r

(14)
$$C_{n;r} = \frac{1}{r} \sum_{j=0}^{r-1} f(t_j) \exp\left[-i \, n \, \frac{2\pi}{r} \, j\right] ,$$

(15)
$$D_{n;r}(t) = A_{n;r} \cos\left(n \, \frac{2\pi}{\theta} \, t\right) + B_{n;r} \sin\left(n \, \frac{2\pi}{\theta} \, t\right) .$$

(i) Exact Interpolation with Dirichlet kernel: for r odd,

(16)
$$f(t) = \sum_{n=0}^{(r-1)/2} D_{n;r}(t)$$

$$= \sum_{n=-(r-1)/2}^{(r-1)/2} C_{n;r} \exp\left[i \, n \, \frac{2\pi}{\theta} \, t\right] .$$

(ii) Exact Interpolation with modified Dirichlet kernel: for r even,

(17)
$$f(t) = A_{0;r} + \sum_{n=0}^{(r/2)-1} D_{n;r}(t)$$

$$+ A_{r/2;r} \cos\left(\frac{r\pi}{\theta} \, t\right) .$$

(iii) Exact Interpolation with Fejèr kernel:

(18)
$$f(t) = \sum_{n=0}^{r-1} \left(1 - \frac{n}{r}\right) D_{n;r}(t)$$

$$= \sum_{n=-r}^{r} \left(1 - \frac{n}{r}\right) C_{n;r} \exp\left[i \, n \, \frac{2\pi}{\theta} \, t\right] .$$

(iv) Exact Interpolation with Tukey kernel:

(19)
$$f(t) = \sum_{n=0}^{r} \frac{1}{2} \left\{1 + \cos \frac{\pi n}{r}\right\} D_{n;r}(t)$$

(v) Exact Interpolation with Parzen kernel:

$$(20)\ f(t) = \sum_{n=0}^{r} \left\{ 1 - 6\left(\frac{n}{2r}\right)^2 + 6\left(\frac{n}{2r}\right)^3 \right\} D_{n;r}(t) + \sum_{n=r+1}^{2r} 2\left\{ 1 - \frac{n}{2r} \right\}^3 D_{n;r}(t)$$

References

R.W. Blackman and J.W. Tukey, The Measurement of Power Spectra, from the point of view of communications engineering. Dover: New York, 1958.

R.W. Hamming, Numerical Methods for Scientists and Engineers. McGraw Hill: New York, 1962.

C. Lanczos, Applied Analysis. Prentice Hall: Englewood Cliffs, New Jersey, 1956.

E. Parzen, "On Statistical Spectral Analysis," to be published by the American Mathematical Society in the Proceedings of a Symposium on Stochastic Processes, to be held in New York, April 30-May 2, 1963.

Table 4A: Harmonic Approximations and their Windows

Coefficient Window $k_M(n)$	Amplitude Window $K_M(z)$	$K_M(0)$	First right hand zero at z =
I. Truncated Fourier Series			
1, n = 0, 1, ... , M.	$D_M(z) = \dfrac{\sin\left(M + \frac{1}{2}\right)z}{\sin \frac{1}{2} z}$	$2M + 1$	$\pi / \left(M + \frac{1}{2}\right)$
II. Modified Truncated Fourier Series	$D_M^*(z) = \dfrac{\sin Mz}{\tan \frac{1}{2} z}$	$2M$	π/M
1, n = 0, 1, ... , M-1;			
$\frac{1}{2}$, n = M.			
III. Fejer means	$F_M(z) = \dfrac{1}{M+1}\left\{ \dfrac{\sin(M+1)\frac{1}{2} z}{\sin \frac{1}{2} z}\right\}^2$	$M + 1$	$2\pi / (M + 1)$
$1 - \frac{|n|}{M+1}$, n=0,1,...,M.			

Footnote: $k_M(0) = 1$, $k_M(-n) = k_M(n)$, $k_M(n) = \frac{1}{2\pi}\int_0^{2\pi} \cos nz\, K_M(z)\, dz$; $K_M(z) = k_M(0) + 2\sum_{j=1}^M k_M(j) \cos jz = \sum_{j=-M}^M k_M(j) \cos jz$

True Window is $\frac{1}{\theta} K_M\left(\frac{2\pi}{\theta} t\right)$ since $f_{M,k}(t) = \int_0^\theta \frac{1}{\theta} K_M\left(\frac{2\pi}{\theta}(x - t)\right) r(x)\, dx$.

Table 4A: Harmonic Approximations and their Windows (continued)

Coefficient Window $k_N(n)$	Amplitude Window $K_N(z)$	$K_N(0)$	First right hand zero at $z =$

IV. Tukey means

$\frac{1}{2}\left(1 + \cos\frac{\pi n}{N}\right)$, n=0,1,...,N-1;

$T_N(z) = \frac{1}{2}D_N^*(z) + \frac{1}{4}D_N^*\left(z + \frac{\pi}{N}\right) + \frac{1}{4}D_N^*\left(z - \frac{\pi}{N}\right)$

$= \sin Nz\left\{\frac{1}{2}\cot\frac{1}{2}z - \frac{1}{4}\cot\frac{1}{2}\left(z + \frac{\pi}{N}\right)\right.$

$\left. - \frac{1}{4}\cot\frac{1}{2}\left(z - \frac{\pi}{N}\right)\right\}$

$K_N(0) = N$

First right hand zero at $2\pi/N$

V. Lanczos means

$\frac{\sin(\pi n/N)}{\pi n/N}$

$L_N(z) = \frac{N}{2\pi}\int_{-\pi/N}^{\pi/N} D_N(z+u)\,du$

$\frac{N}{2\pi}\int_{-\pi/N}^{\pi/N} D_N(u)\,du = \frac{N}{2\pi}\int_{-\pi/N}^{\pi/N}\frac{\sin Nu}{\frac{1}{2}u}\,du$

$= \frac{2N}{\pi}\int_0^\pi \frac{\sin u}{u}\,du$

$= (1.18)N$

VI. Parzen means

$1-6\left(\frac{n}{N}\right)^2+6\left(\frac{n}{N}\right)^3$, n=0,1,...,$\frac{N}{2}$.

$2\left(1 - \frac{n}{N}\right)^3$, n=$\frac{N}{2}+1,...,N$.

$P_N(z) = \frac{3}{4N^3}\left\{\frac{\sin\frac{1}{4}Nz}{\frac{1}{2}\sin\frac{z}{2}}\right\}^4\left\{1 - \frac{2}{3}\left(\sin\frac{z}{2}\right)^2\right\}$

for N even

$\frac{3}{4}N$

$4\pi/N$

Table 4B: Window Generators

$k(u)$	$k_M(n) = k\left(\dfrac{u}{M_M}\right)$	$K(z) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-iuz}\, k(u)\, du$	Approximation $K_M(z) = 2\pi M\, K(Mz)$	$K_M(0)$
I. Dirichlet $k_D(u) = 1,\ \|u\| \leq 1$ $= 0,\ \|u\| > 0$	If $M = N$, $k_M(n) = 1$, $n = 0, \pm 1, \ldots, \pm N$	$\dfrac{1}{\pi}\dfrac{\sin z}{z}$	$\dfrac{\sin Nz}{z/2}$	$2N$
II. Modified Dirichlet $k_D^{*}(u) = 1,\ \|u\| < 1$ $= \tfrac{1}{2},\ \|u\| = 1$ $= 0,\ \|u\| > 1$	If $M = N$, $k_M(n)=1,\ n=0,\ \pm 1,\ldots,\pm(N-1)$, $= \tfrac{1}{2},\ n = N$	$\dfrac{1}{\pi}\dfrac{\sin z}{z}$	$\dfrac{\sin Nz}{z/2}$	$2N$
III. Fejer $k_F(u) = 1 - \|u\|,\ \|u\| \leq 1$ $= 0,\ \|u\| > 0$	If $M = N + 1$, $k_M(n) = 1 - \dfrac{\|n\|}{N+1},\ n=0,\ldots,\pm N$	$\dfrac{1}{2\pi}\left\{\dfrac{\sin z/2}{z/2}\right\}^{2}$	$\dfrac{1}{N}\left\{\dfrac{\sin Nz/2}{z/2}\right\}^{2}$	N

| $k(u)$ | $k_M(n) = k\left(\frac{u}{M_N}\right)$ | $K(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuz}\, k(u)\, du$ | Approximation $K_M(z) = 2\pi M\, K(Mz)$ | $K_M(0)$ |

IV. Tukey

If $M = N + \frac{1}{2}$,

$k_T(u) = \frac{1}{2}(1 + \cos \pi u)$, $|u| \le 1$

$k_M(n) = \frac{1}{2}\left(1 + \cos \frac{\pi n}{N + \frac{1}{2}}\right)$

$\dfrac{1}{2\pi}\, \dfrac{\sin z}{z}\, \dfrac{\pi^2}{\pi^2 - z^2}$

$\dfrac{\sin Mz}{z}\, \dfrac{\pi^2}{\pi^2 - (Mz)^2}$

M

V. Lanczos

If $M = N$,

$k_L(u) = \dfrac{\sin \pi u}{\pi u}$, $|u| \le 1$.

$= 0$, $|u| > 1$

$k_M(n) = \dfrac{\sin \frac{\pi n}{N}}{\frac{\pi n}{N}}$

$\dfrac{1}{2\pi^2}\left\{ Si\,(z+\pi) - Si\,(z-\pi)\right\}$,

$Si\,(x) = \displaystyle\int_0^x \frac{\sin u}{u}\, du$

$= \displaystyle\int_0^1 \frac{\sin xu}{u}\, du$,

$Si\,(-x) = -Si(x)$

$\dfrac{M}{\pi}\left\{ Si\,(Mz+\pi) - Si\,(Mz-\pi)\right\}$

$\dfrac{2M}{\pi}\, Si\,(\pi) = M\, \dfrac{3.7}{\pi}$

$= (1.18)\, M$

VI. Parzen

$k_P(u) = 1 - 6u^2 + 6|u|^3$, $|u| \le \frac{1}{2}$

$= 2(1 - |u|)^3$, $\frac{1}{2} \le |u| \le 1$

$\dfrac{3}{8\pi}\left\{\dfrac{\sin z/4}{z/4}\right\}^4$

$\dfrac{3}{4M^3}\left\{\dfrac{\sin Mz/4}{z/4}\right\}^4$

$\dfrac{3}{4}M$
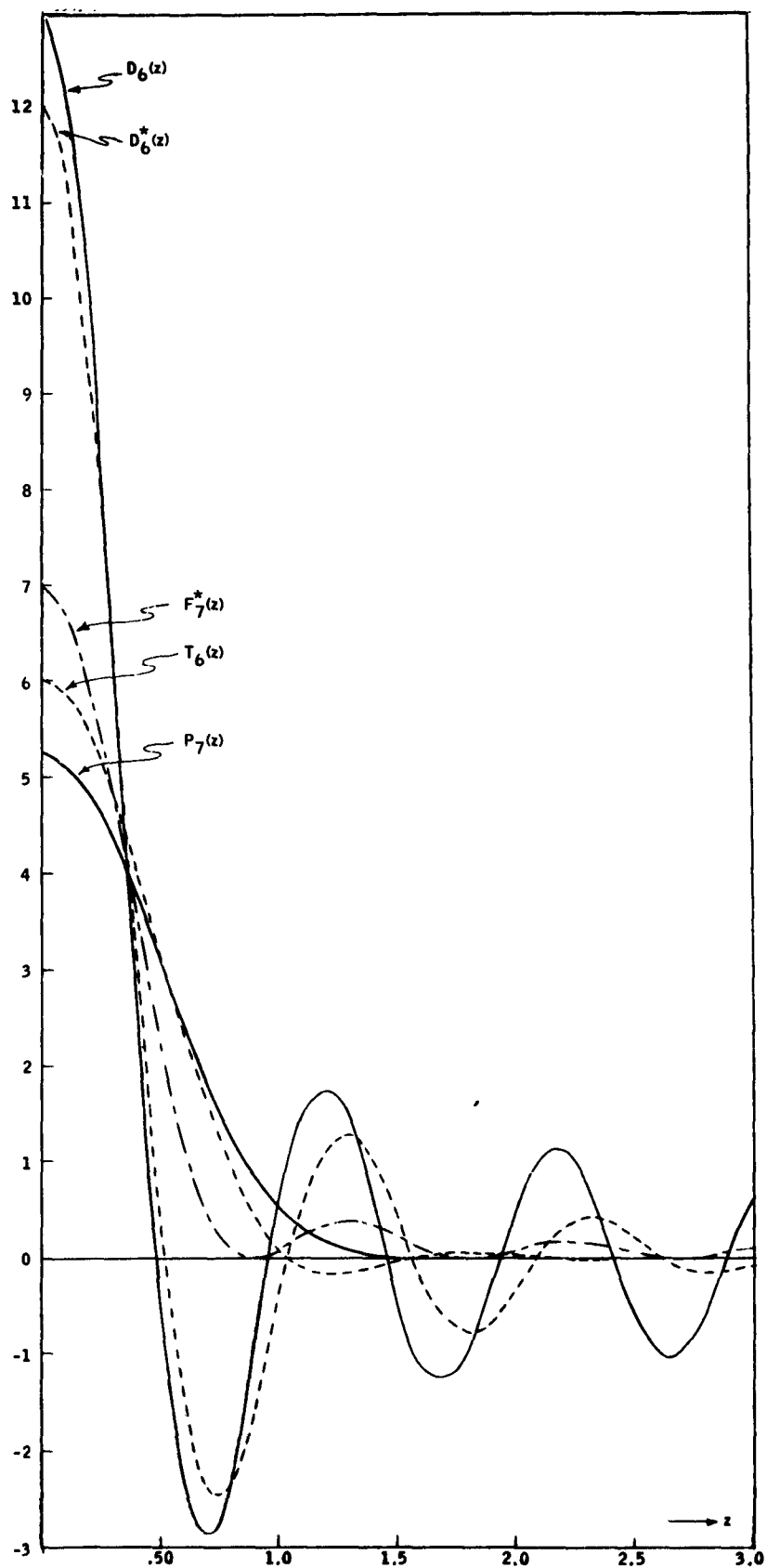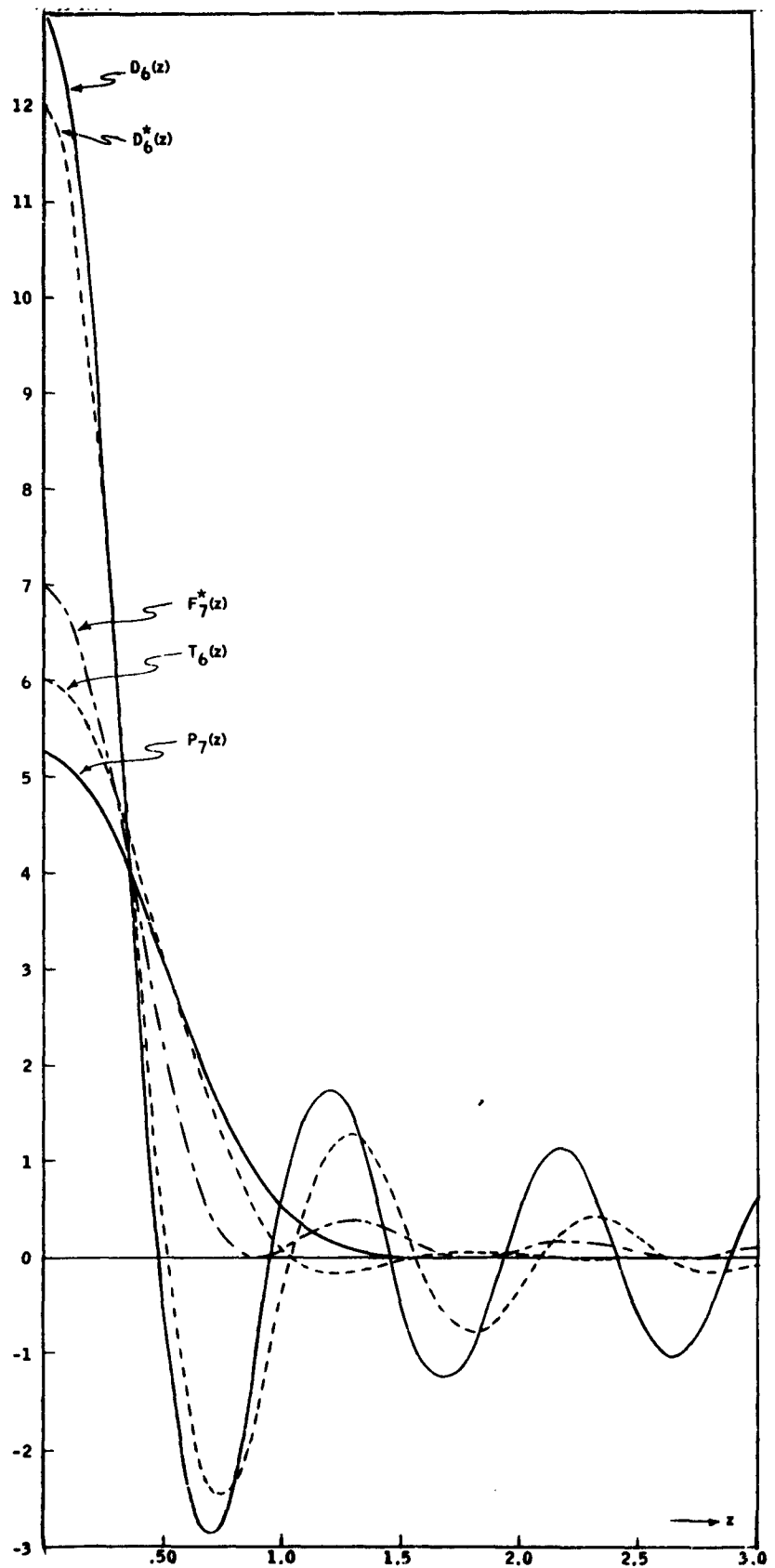
Figure 3A. Windows.

Figure 3A. Windows.
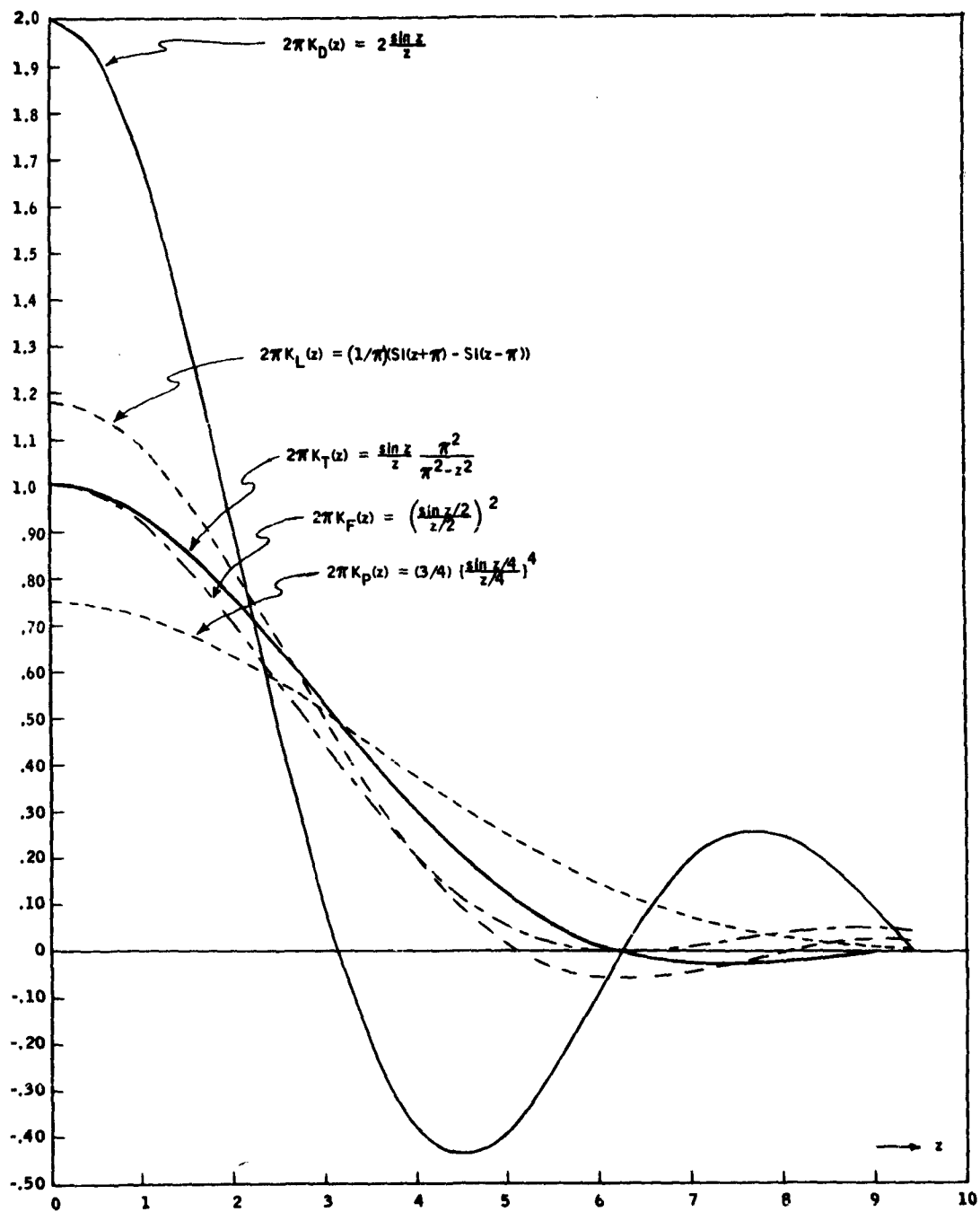
Figure 4A.   Amplitude Window Generators.
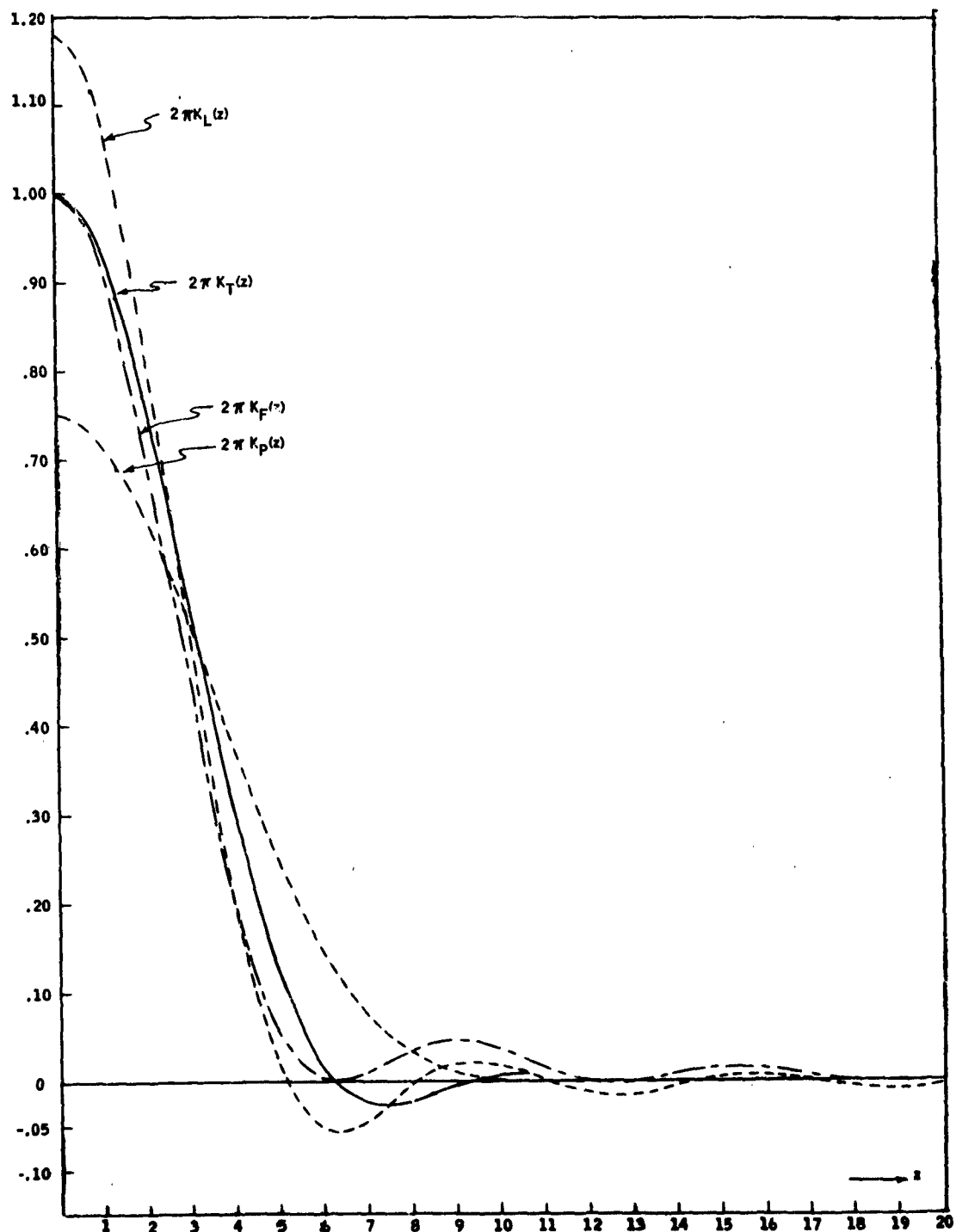
Figure 4A'.   Amplitude window generators;   Figure 4A redrawn on an enlarged scale.
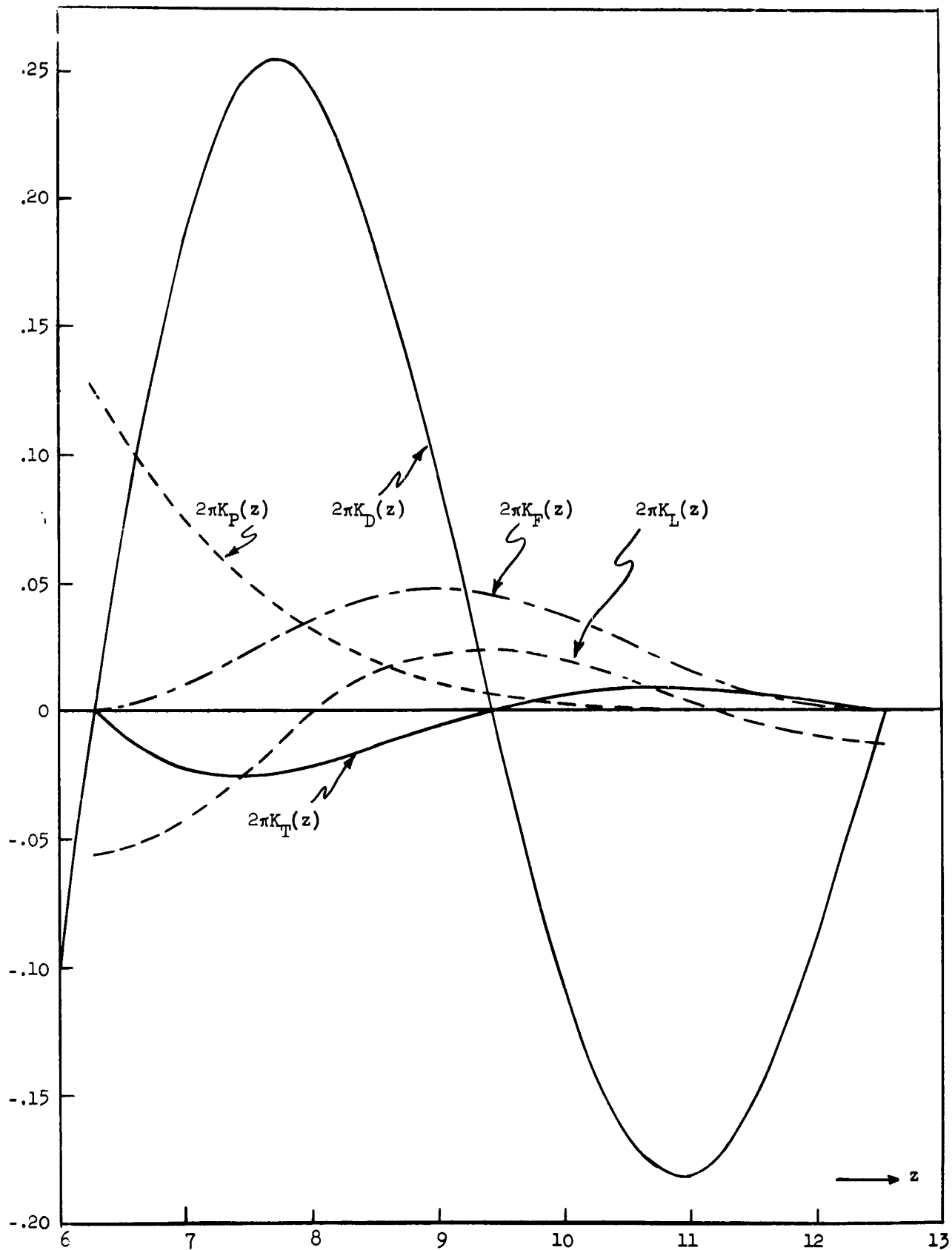
Figure 4A". Continued on an enlarged scale.
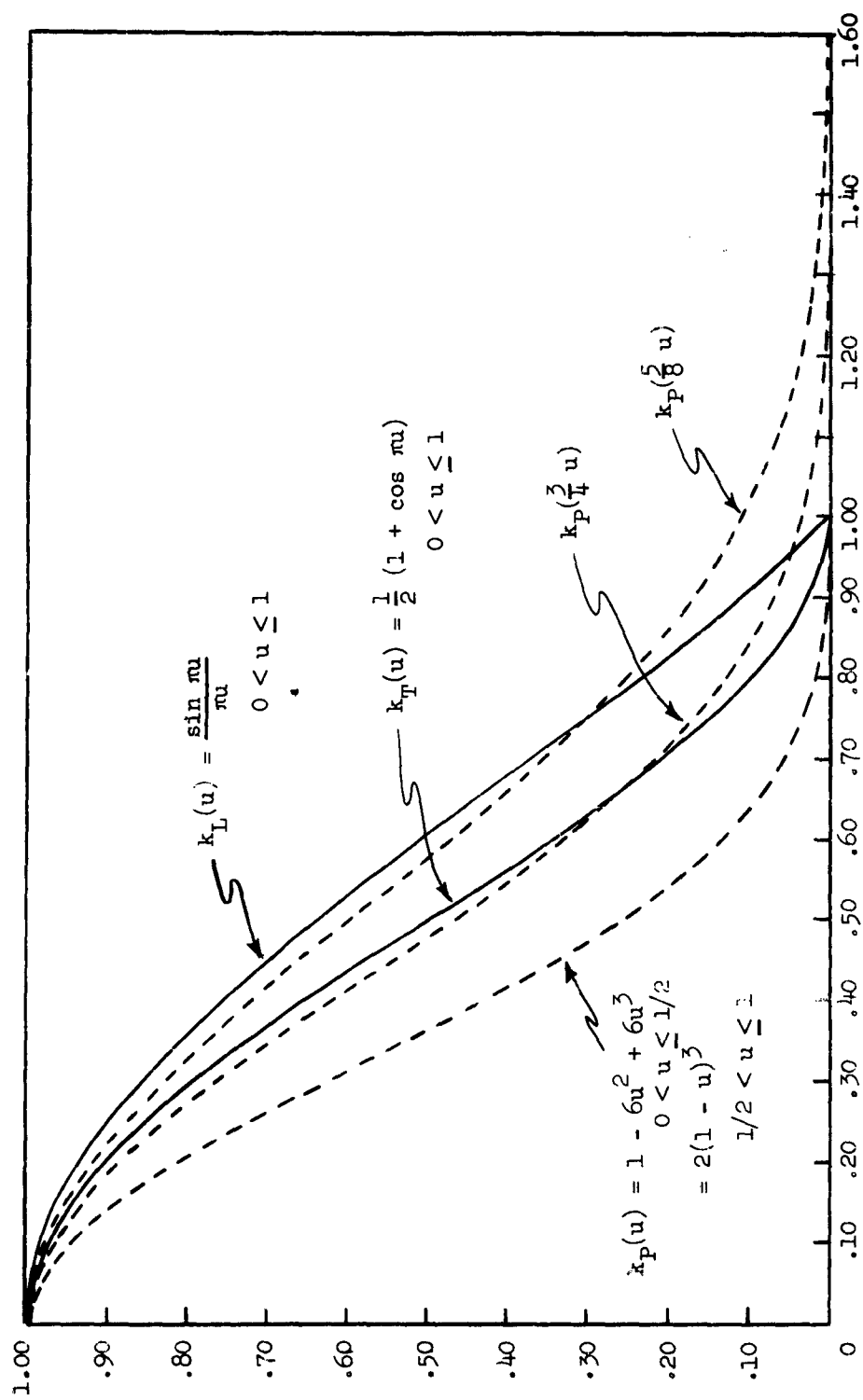
Figure 4B. Coefficient Window Generators
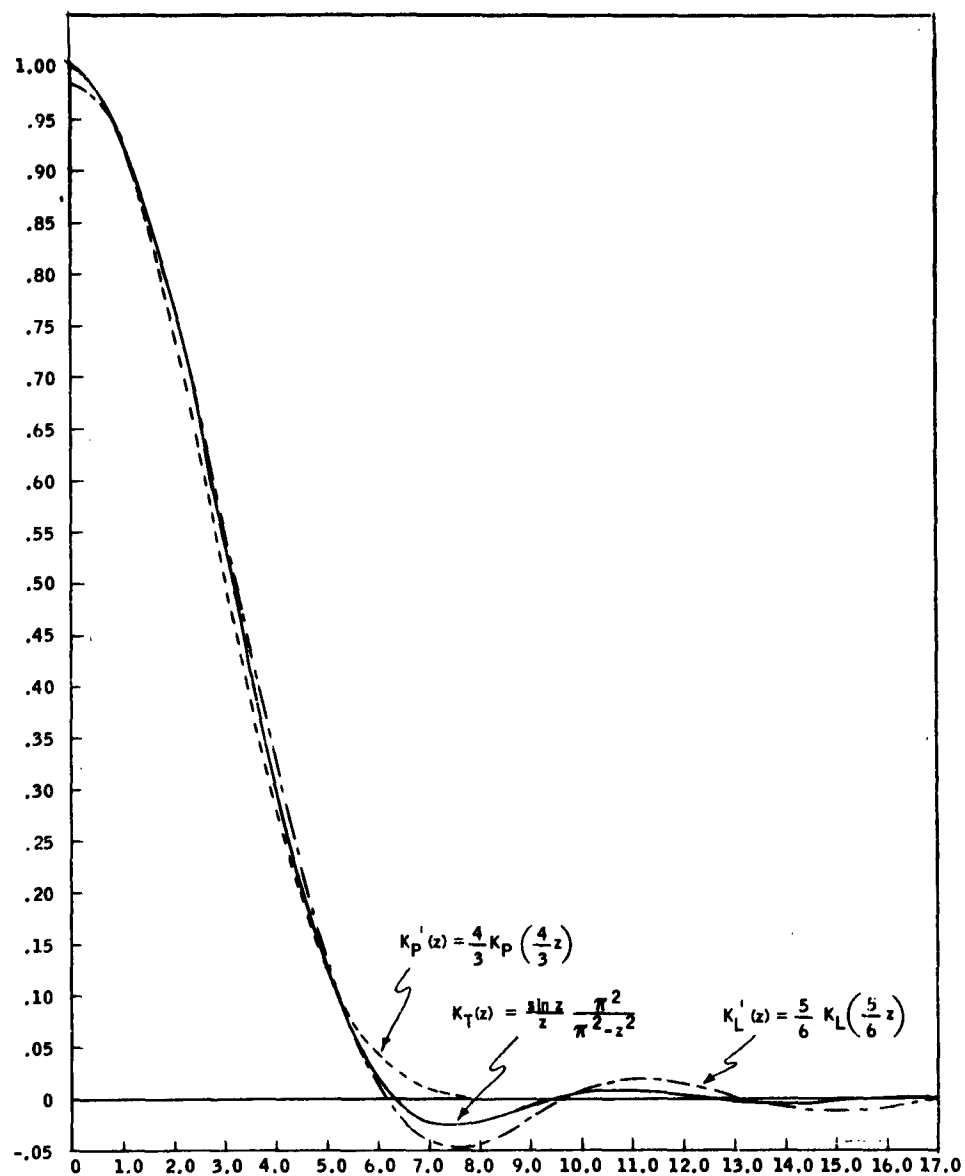
Figure 4C. Some windows normalized to have approximately same height at center frequency.
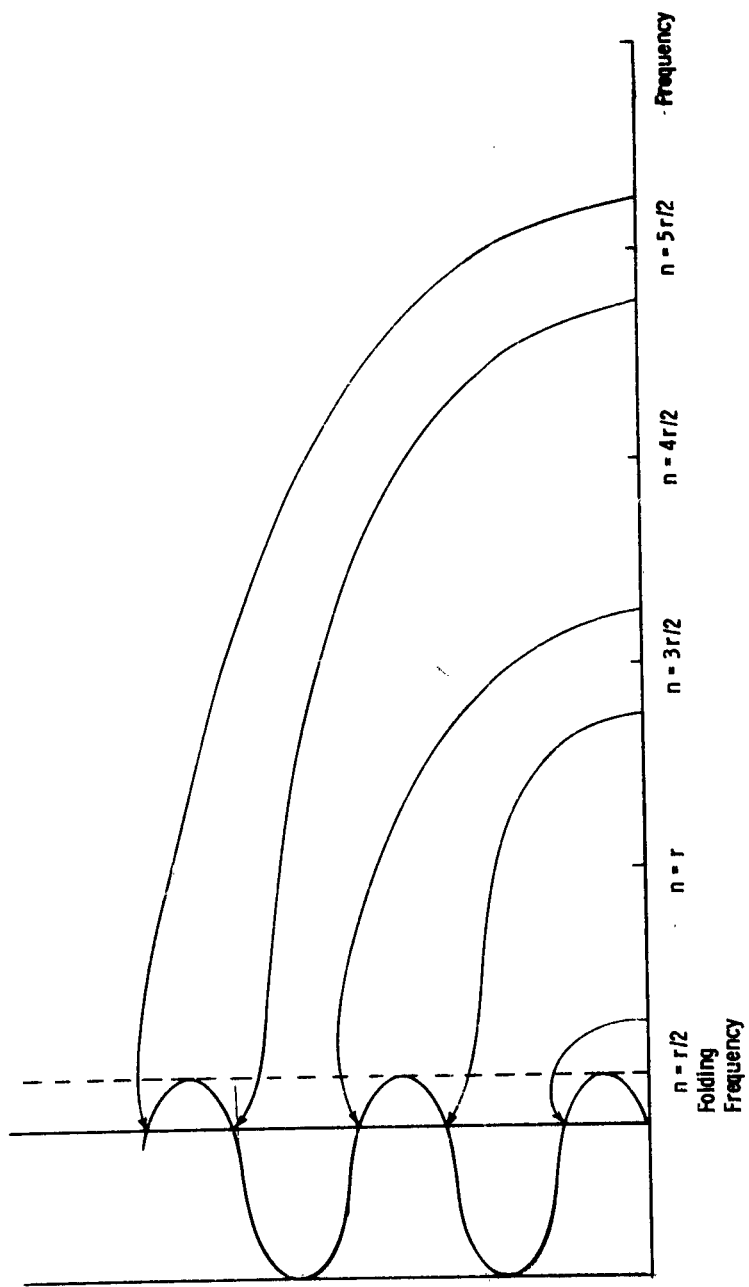
Figure 5A. Pictorial representation of aliasing of Frequencies.

Armed Services Technical
  Information Agency
Arlington Hall Station
Arlington 12, Virginia ... 10

Bureau of Supplies and Accounts
  Code OW
Department of the Navy
Washington 25, D. C. ... 1

Head, Logistics and Mathematical
  Statistics Branch
Code 436
Office of Naval Research
Washington 25, D. C. ... 3

Commanding Officer,
Office of Naval Research Branch Office
346 Broadway
New York 13, New York
Attn: J. Laderman ... 1

Commanding Officer,
Office of Naval Research Branch Office
1030 East Green Street
Pasadena 1, California (Dr. A.R. Laufer)

Commanding Officer,
U. S. Army Signal Electronic Research
P. O. Box 205 - Major Ademetto
Mountain View, California

Commanding Officer
U. S. Navy Powder Factory
  Attn: F. Freshman, R & D
Indianhead, Maryland

Director, Naval Research Lab.
  Attn: Tech. Infor. Officer
Washington 25, D. C.

Director,
National Security Agency
  Attn: REMP-1
Fort George G. Meade, Maryland

Dept. of Mathematical Statistics
University of North Carolina
Chapel Hill, North Carolina

Department of Statistics
University of California
Berkeley 4, California

Department of Statistics
Michigan State University
East Lansing, Michigan

Library,
Institute for Defense Analyses
Communications Research Division
Von Neumann Hall
Princeton, New Jersey

Library
Department of Industrial Engr.
  & Operations Research
College of Engr.
New York University
University Heights
New York 53, New York

Office of the Ass't Naval
  Attache for Research
American Embassy
Navy No. 100, Fleet P. O.
New York, New York

Scientific Section
Office of Naval Research
1000 Geary Street
San Francisco 9, California

Statistical Engineering Lab.
National Bureau of Standards
Washington 25, D. C.

Statistical Laboratory
University of Washington
Seattle 5, Washington

Superintendent,
U. S. Navy Postgraduate School
  Attn: Library
Monterey, California

U. S. Naval Avionics Facility
  Attn: Library
Indianapolis 18, Indiana

Dr. Stephen G. Allen
Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California ... 10

Professor T. W. Anderson
Dept. of Mathematical Statistics
Columbia University
New York 27, New York ... 1

Professor Fred C. Andrews
Department of Mathematics
University of Oregon
Eugene, Oregon

Professor Robert Bechhofer
Dept. of Industrial and Eng. Admin.
Sibley School of Mechanical Eng.
Cornell University
Ithaca, New York ... 1

Mr. J. Berkowitz
Quality Control Supervisor
NORDEN Division of United Aircraft Corp.
Ketay Department - Helen Street
Norwalk, Connecticut

Dr. Julian H. Bigelow
School of Mathematics
Institute for Advanced Study
Princeton, New Jersey ... 1

Professor Z. W. Birnbaum
Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle 5, Washington ... 1

Professor David Blackwell
Department of Mathematical Sciences
University of California
Berkeley 4, California ... 6

Dr. Julius R. Blum
Sandia Corporation
Albuquerque, New Mexico ... 2

Dr. Paul Blunk
Box 532
Fair Oaks, California ... 1

Dr. Charles Boll
Aerospace Corp. F-2813
P. O. Box 95085
Los Angeles 45, California ... 1

Professor G. E. P. Box
Department of Statistics
University of Wisconsin
Madison, Wisconsin ... 1

Professor Ralph A. Bradley
Department of Statistics
Florida State University
Tallahassee, Florida ... 1

Professor W. G. Cochran
Department of Statistics
Harvard University
2 Divinity Avenue, Room 313
Cambridge 38, Massachusetts ... 1

Professor Lee Cronbach
Bureau of Education Research
1007 S. Wright
Champaign, Illinois ... 2

Professor Cyrus Derman
Dept. of Industrial Engineering
Columbia University
New York 27, New York ... 2

Dr. Joseph Daly
Bureau of the Census
Washington 25, D. C. ... 1

Dr. Francis Dresch
Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California ... 1

Mr. John A. Dutton
Department of Meteorology
University of Wisconsin
Madison, Wisconsin ... 1

Professor Meyer Dwass
Department of Mathematics
Northwestern University
Evanston, Illinois ... 1

Professor W. T. Federer
Cornell University
Department of Plant Breeding
Biometrics Unit
Ithaca, New York ... 1

George Fishman
Logistics Department
Rand Corporation
1700 Main Street
Santa Monica, California ... 1

Dr. Merrill M. Flood
Mental Health Research Institute
205 North Forest Avenue
Ann Arbor, Michigan ... 1

Dr. Donald P. Gaver
Westinghouse Research Labs.
Beulah Rd. - Churchill Boro.
Pittsburgh 35, Pennsylvania ... 1

Mr. Murray A. Geisler
Logistics Section
The RAND Corporation
1700 Main Street
Santa Monica, California ... 1

Professor H. P. Goode
Dept. of Industrial and Engineering
  Administration
Cornell University
Ithaca, New York ... 1

Professor E. J. Gumbel
Industrial Engr. Department
409 Engineering Building
Columbia University
New York 27, New York ... 1

Professor Donald Guthrie, Jr.
Stanford Research Institute
333 Ravenswood Avenue
Menlo Park, California ... 1

Professor Bernard Harris
Department of Mathematics
The University of Nebraska
Lincoln, Nebraska ... 1

Dr. Theodore E. Harris
The RAND Corporation
1700 Main Street
Santa Monica, California ... 1

Leon H. Herbach
Dept. of Industrial Engineering
and Operations Research
College of Engineering
New York University
New York 53, New York ... 1

Professor W. Hirsch
Institute of Mathematical Sciences
New York University
New York 3, New York ... 1

Professor Harold Hotelling
Associate Director,
Institute of Statistics
University of North Carolina
Chapel Hill, North Carolina ... 1

Professor Stanley Isaacson
4706 Lakeview Drive
Des Moines 12, Iowa ... 1

Professor Oscar Kempthorne
Statistics Laboratory
Iowa State College
Ames, Iowa ... 1

Professor Solomon Kullback
Department of Statistics
George Washington University
Washington 7, D. C. ... 1

Dr. Eugene Lukacs
Department of Mathematics
Catholic University
Washington 17, D. C. ... 1

Professor G. W. McElrath
Department of Mechanical Engineering
University of Minnesota
Minneapolis 14, Minnesota ... 1